DOCUMENT RESUME

ED 326 447 SE 051 810

TITLE Three-Year Sequence for High School Mathematics, Course II.

INSTITUTION New York State Education Dept., Albany. Bureau of Curriculum Development.; New York State Education Dept., Albany. Bureau of Mathematics Education.

PUB DATE 89

NOTE 80p.; For Course III, see ED 324 236.

AVAILABLE FROM Publications Sales Desk, New York State Education Department, Room 171 EBA, Wasnington Avenue, Albany, NY 12234.

PUB TYPE Guides - Classroom Use - Guides (For Teachers) (052)

EDRS PRICE MF01/PC04 Plus Postage.

DESCRIPTORS Activity Units; Algebra; Calculators; Computer Assisted Instruction; Disabilities; Enrichment; Geometry; High Schools; *Instructional Materials; Interdisciplinary Approach; *Learning Activities; Mathematical Logic; Mathematics Curriculum; Mathematics Education; Mathematics Instruction; Probability; *Problem Solving; Resource Materials; *Secondary School Mathematics; State Curriculum Guides; *Teaching Methods; Teaching Skills; Thinking Skills; Units of Study

ABSTRACT

The purpose of this revised syllabus/guide is to help teachers and administrators plan and implement Course II in the the State Three-Year Sequence for High School Mathematics for Regents credit. Course II builds upon the state mathematics curriculum outlined in the department's mathematics K-6, Grade 7, Grade 8, and Course I publications. Included in this guide are the major and minor changes from the 1977 edition, the role of problem solving, the course II outline, an evaluation, and resources for teachers. Appended are a statement of Regents' goals from 1984 and a discussion of special needs of handicapped students. The course outline includes the topics: (1) "Logic"; (2) "Mathematical Structure and Related Algebraic Skills"; (3) "Euclidean Geometry"; (4) "Analytic Geometry"; (5) "Equations and Inequalities"; and (6) "Probability and Combinations." Each section contains examples and sample problems related to that section. (KR)

**********
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
**********

# Three-Year Sequence for High School Mathemati[illegible]

## Course II

PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY
Robert M. Trombly
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC).

ED326447

SE051810

The University of the State of New York
The State Education Department
Bureau of Curriculum Development
Albany, New York 12234

THE STATE EDUCATION DEPARTMENT

TO [illegible]

FROM [illegible]

[illegible] is designed to aid schools in planning a course of study for students in Course II. This *Syllabus Guide* continues the development of the program begun in Course I. It is intended to serve as the basis for local activities such as:

Development of local courses of study to meet the standards of the *Syllabus Guide*.

Selection and acquisition of support materials.

Articulation of local offerings from grade to grade or building to building.

Evaluation of student performance.

As is the policy of the State Education Department, this publication will be periodically reviewed. Your comments on it are welcome and should be addressed to:

*Bureau of Mathematics Education*
*New York State Education Department*
*Albany, New York 12234*

*Additional copies of this publication can be obtained by submitting requests in writing on school stationery to the:*

*Publications Sales Desk*
*New York State Education Department*
*Room 171 EBA*
*Washington Avenue*
*Albany, New York 12234*

# Three-Year Sequence for High School Mathematics

# Course II

Reprinted 1989

The University of the State of New York
The State Education Department
Bureau of Mathematics Education
Bureau of Curriculum Development
Albany, New York 12234

**THE UNIVERSITY OF THE STATE OF NEW YORK**

**Regents of The University**

Martin C. Barell, *Chancellor*, B.A., J.A., LL.B. — Muttontown
R. Carlos Carballada, *Vice Chancellor*, B.S. — Rochester
Willard A. Genrich, LL.B. — Buffalo
Emlyn I. Griffith, A.B., J.D. — Rome
[illegible] — [illegible]
[illegible] — Chestertown
[illegible] — [illegible]
[illegible] — [illegible]
[illegible] — [illegible]
[illegible] — [illegible]
[illegible] — [illegible]
[illegible] — [illegible]
[illegible] — [illegible]
[illegible] — [illegible]
[illegible] — [illegible]
[illegible] — Johnson City
[illegible] — [illegible]
[illegible] — Rochester
Gerald C. Lubin, B.A., M.B. — Staten Island

**President of The University and Commissioner of Education**
Thomas Sobol

**Executive Deputy Commissioner of Education**
Thomas E. Sheldon

**Deputy Commissioner for Elementary and Secondary Education**
Lionel R. Meno

**Assistant Commissioner for General and Occupational Education**
*Lorraine Sheinkopf*

**Director, Division of General Education**
*Charles J. Taupin*

**Chief, Bureau of Mathematics Education**
*Fredric Paul*

**Assistant Commissioner for Elementary and Secondary Education**
**Planning, Testing and Technological Services**
*Judith S. Mounty*

**Director, Division for Program Development**
Edward T. Eldon

**Chief, Bureau of Curriculum Development**
Vacant

The State Education Department does not discriminate on the basis of age, color, religion, creed, disability, marital status, veteran status, national origin, race, or sex in the educational programs and activities which it operates. Inquiries concerning this policy of equal opportunity and affirmative action should be referred to the Department's Affirmative Action Officer, Education Building, Albany, NY 12234. Phone (518) 474-1265.

# Foreword

### Purpose

The purpose of this *Syllabus/Guide* is to help teachers and administrators plan and implement Course II in the State Three-Year Sequence for High School Mathematics for Regents credit. Course II builds upon the State mathematics curriculum outlined in the Department's mathematics *K-6*, *Grade 7*, *Grade 8*, and *Course I* publications.

### Audience

The audience for this publication includes teachers of courses in the Three-Year Sequence for High School Mathematics; high school mathematics chairpersons; coordinators and curriculum developers; and administrators.

### Development

This *Syllabus/Guide* has been developed by the Department over a period of several years, and reflects contemporary thinking about the mathematics curriculum for the high school.

As such, this document addresses a 1985 recommendation of the College Entrance Examination Board: the high school mathematics curriculum should be organized so that the topics of computing, statistics, algebra, geometry, and functions are integrated into a three-year series of courses. Also, an important feature of this *Syllabus/Guide* is attention to problem solving, per the recommendations of the National Council of Teachers of Mathematics' "Agenda for Action: Recommendations for School Mathematics of the '80s."

The initial step in preparing this *Syllabus/Guide* to implement the above recommendations was the appointment, in May 1984, of an ad hoc committee, by the Bureau of Mathematics Education. The charge of this committee was to advise the Department regarding direction for the mathematics curriculum in the high school. The committee, which met in June 1984, consisted of the following:

Steven Adrian, Mahopac High School

Janet Burt, Clinton High School

Walter Callahan, Shenendehowa Middle School, Elnora; President, Association of Mathematics Teachers of New York State, 1983-84

Bernard Fleishman, Professor of Mathematics, Rensselaer Polytechnic Institute, Troy

Joel Friedberg, Herbert H. Lehman High School, New York City; President, Mathematics Chairmen's Association of New York City, 1984-86

Jack Graver, Professor of Mathematics, Syracuse University

Meyer Jordan, Professor of Mathematics, Brooklyn College

Irwin Kaufman, Director of Mathematics, New York City

Edward Keenan, The Wheatley School, Old Westbury

Edith Luchins, Professor of Mathematics, Rensselaer Polytechnic Institute, Troy

Timothy McMullin, Vestal Senior High School

Lester Rubenfeld, Professor of Mathematics, Rensselaer Polytechnic Institute

Sister Ann Xavier, Nazareth Academy, Rochester

In June 1984, the State Education Department announced that the integrated three-year sequence would become the official Statewide program for Regents high school mathematics beginning in September 1987. A Revision Advisory Committee was appointed in July 1984. This committee has since met regularly to review draft materials, and to provide input to the Department. It has included the following:

Janet Burt, Clinton High School

David Call, Fort Ann Central School

Roberta Eisenberg, Beach Channel High School, New York City

Pierre Fabinski, Christopher Columbus High School, New York City

Grahme Fischer, Technical Problem Solvers, Hauppauge

Anne Flansburg, Lafayette High School, Buffalo

Bernard Fleishman, Rensselaer Polytechnic Institute, Troy

Robert Frankel, Tottenville High School, New York City; President, Mathematics Chairmen's Association of New York City, 1986-88

Joel Friedberg, Herbert H. Lehman High School, New York City; President, Mathematics Chairmen's Association of New York City, 1984-86

Jack Graver, Syracuse University

Richard Klutch, Hunter College High School, New York City

Sharon Lanza, Grover Cleveland High School, Buffalo

Edith Luchins, Rensselaer Polytechnic Institute, Troy

Shelley Mora, Clara Barton High School, New York City

Fred Pedersen, Herricks High School, New Hyde Park

Lester Rubenfeld, Rensselaer Polytechnic Institute, Troy

George Stevens, Lansing High School

Donald Thorpe, Vestal Senior High School

Sister Ann Xavier, Nazareth Academy, Rochester

Using the recommendations of the Advisory Committee, Sister Ann Xavier, Janet Burt, and Richard Klutch organized and wrote a *Course II* draft Syllabus/Guide. Eleven schools/school districts field-tested it during the 1986-87 school year:

Buffalo City School District

Colonie Central High School, Albany

Dobbs Ferry High School

The Wheatley School, East Williston

Fort Ann Central School

Herricks High School

Mahopac High School

New York City
- Christopher Columbus High School
- Clara Barton High School
- Herbert H. Lehman High School
- Tottenville High School

Rochester City School District

Nottingham High School, Syracuse

Vestal Senior High School

This *Syllabus/Guide* was prepared under the direction of Ben Lindeman and Lee Negus, Associates in the Bureau of Mathematics Education, with the assistance of Robert Carruthers, Associate in the Bureau of Curriculum Development. Staff from the Office for the Education of Children with Handicapping Conditions reviewed the manuscript in the light of the needs of students with handicapping conditions. Michael Moon, Associate in the Division of Civil Rights and Intercultural Relations, reviewed the manuscript for equity concerns. Marjorie Edelstein, Christopher Columbus High School, New York City, reviewed the manuscript for environmental concerns. Mary Theresa Southworth, Consultant for the Bureau of Curriculum Development, assisted in the revising and editing of the final manuscript.

# Contents


FOREWORD ..... v

INTRODUCTION ..... 1

GOALS AND LEARNER OUTCOMES ..... 4

THE ROLE OF PROBLEM SOLVING ..... 6

COURSE II OUTLINE ..... 11

I. Logic ..... 15

II. Mathematical Structure and Related Algebraic Skills ..... 23

III. Euclidean Geometry ..... 35

IV. Analytic Geometry ..... 45

V. Equations and Inequalities ..... 57

VI. Probability and Combinatorics ..... 65

EVALUATION ..... 71

RESOURCES FOR TEACHERS ..... 73

APPENDIX A—STATEMENT OF REGENTS GOALS—1984 ..... 75

APPENDIX B—STUDENTS WITH HANDICAPPING CONDITIONS ..... 77

# Introduction

*Overview*. This *Syllabus/Guide* presents an outline and some teaching notes for Course II of the Three-year Sequence for High School Mathematics program of New York State. First, the topics included in the program are spiraled and sequenced throughout the three years so that the students review and build on previous learning. Second, the branches of mathematics, including the traditional topics of algebra, geometry, and trigonometry, and the additional topics of logic, probability, statistics, and transformational geometry, are integrated. Suggestions for such integration are included throughout the *Syllabus/Guide*. Although the descriptions of topics to be taught are presented here in a traditional unit format for the convenience of teachers and supervisors, teachers should integrate these topics in their classrooms when possible. This integration, however, should be a natural, not a forced, one which benefits the instructional process and the learner outcomes.

*Use of the Syllabus/Guide*. It is expected that the material presented here can be completed by most classes in one school year, usually the tenth grade. Some students will complete the course in less than one year; others will require more than one year. Likewise, although the course is written with the tenth grade student in mind, some students may begin the material either in an earlier or a later year. It is expected that individual teachers will consider their own students in making judgments concerning time allotments, aspects of the material to be stressed, the degree of rigor employed with any topic, and extra material to be presented. It should be noted, as in Course I, that teachers need not present the topics in the order given in the outline.

*Changes from 1977 edition*. *Major* changes in the Course II syllabus include (1) more specificity for topics, (2) an attempt to infuse problem solving throughout, (3) making optional the formal study of groups and fields (including theorems and applications), and (4) an increase in the extension of algebraic skills and concepts from Course I. *Minor* changes in the syllabus include (1) the transfer of constructions from Course I to Course II, (2) the inclusion of geometric transformations in the coordinate plane, (3) the inclusion of right triangle trigonometry, and (4) the transfer of quantifiers from Course II to Course III.

LOGIC: In Course I, the study of logic provided the students with the foundation for deductive thinking. In Course II, the study of Logic formalizes the laws of inference, and the students are expected to strengthen their thinking skills through an introduction to mathematical proof.

GEOMETRY: The core of Course II is a solid foundation in the development of geometric concepts. Many of the concepts and relationships developed intuitively in Course I should be proven formally in Course II. Major concepts include congruence, similarity, parallelism, and inequalities. Euclidean geometry, analytic geometry, and geometry using transformations are all presented in Course II.

ALGEBRA: The algebra of Course II should review and extend the ideas developed in Course I, and should introduce the students to some basic ideas of mathematical structure. The ability to apply algebraic skills and processes is vital to any further study of mathematics; thus, the students should have ample reinforcement of algebraic techniques, including factoring, operating with fractions, and solving equations. In Course I, the students learned to solve quadratic equations with rational roots by factoring. However, a false impression is created if none of the problems a student encounters involving quadratic equations has irrational solutions. Therefore, in Course II the quadratic formula should be derived and used. Further, the graph of $y = ax^2 + bx + c$ is a reasonable extension of the graphing in Course I as well as the work with locus in Course II.

PROBABILITY AND COMBINATORICS: Permutations and combinations should be studied thoroughly.

They provide the foundation for some good problem solving in probability.

For Course II, in addition to the above content the items listed below should be considered in the implementation of the syllabus.

ESTIMATION AND APPROXIMATION: Estimation and approximation are important strategies used in the elementary and junior high schools and should be continued in Course I and Course II where appropriate. Estimation is often useful in areas such as finding square root, or in probability activities, and should be systematically used when solving problems. Estimation, where appropriate, should be one of the steps in the problem-solving model (See "The Role of Problem Solving", pp. 6 to 9). The students will not use estimation unless they find it useful; therefore, the teacher's role in teaching estimation should be one of modeling and motivating.

COMPUTERS AND CALCULATORS: As in Course I, in Course II there are no formal units on the computer or the calculator. Nonetheless, the use of calculators, computers, and flowcharting is suggested in those areas where such devices can contribute to mathematical understanding, increase the students' participation and enjoyment, or broaden the area from which applications of mathematical ideas can be selected. Individual teachers may wish to give lesser or greater emphasis to the role which these modern tools can play in the study of mathematics.

PROBLEM SOLVING: Problem solving is a major focus of mathematics education. The teacher should make every effort to use problems in daily teaching. In Course II, problems should include proofs (logic, Euclidean, and analytic), application of geometric concepts, application of algebraic skills and techniques, application of the quadratic equation, and right triangle trigonometry. Both routine and non-routine problems should be presented.

UNITS AND DIMENSIONAL ANALYSIS: When presenting problems that involve measurement, the teacher should take care to include necessary units along with the numerical part of the measure. In science, engineering, and some industrial applications, it is necessary to change measurements from one unit to another. Dimensional analysis is a technique used to accomplish such conversions. This technique treats both a number and its unit of measure as factors. "Computations" are usually done with ratios where common units of measure are "divided out" in the same way common number factors are "divided out" or "canceled."

Although some consider this a process which is not mathematically correct, the symbolism has been developed by practitioners in such a way as to yield correct results.

**EXAMPLE 1:**

The area of a trapezoid is 85 $cm^2$, and the bases are 7 cm and 10 cm, respectively. Find the height of the trapezoid.

$$A = 1/2h\,(b_1 + b_2)$$

$$85\ cm^2 = 1/2h\,(7\ cm + 10\ cm)$$

$$170\ cm^2 = h\,(17\ cm)$$

$$h = \frac{170\ cm^2}{17\ cm} = 10\ cm \text{ Ans.}$$

**EXAMPLE 2:**

Convert 8 $cm^2$ to $mm^2$

$$1\ cm^2 = 10\ mm \times 10\ mm = 100\ mm^2$$

$$1 = \frac{100\ mm^2}{1\ cm^2}$$

$$\text{and } 8\ cm^2 \cdot \frac{100\ mm^2}{1\ cm^2} = 800\ mm^2 \quad \text{Ans.}$$

INTERDISCIPLINARY ACTIVITIES: The Regents Action Plan recommends the use of special interdisciplinary projects to show the students that the various subject areas are interrelated. Science, business, the social sciences (including economics), and technology education make extensive application of mathematical skills and concepts. Problems/situations from these areas will show the students the value of mathematics as a tool used in many human endeavors.

Music and art also may be used as vehicles to show the importance of mathematics in the arts. Much music theory is based upon basic principles of mathematics. Symmetry, tessellations, and form play significant roles in various forms of art.

The role of pure mathematics cannot be overlooked when looking at the interdisciplinary aspects of mathematics. Major scientific theories of the 19th and 20th centuries were able to be developed because the mathematics for them had been created before anyone saw their possible applications to science. For example, Einstein's general theory of relativity would not have been possible without the geometry of Riemann,invented in 1854, or without the theory of invariance developed by Cayley, Sylvester, and other 19th century mathe-

maticians. Many such examples could be given, to show, as Gauss stated, "Mathematics is Queen of the Sciences."

**LANGUAGE AND NOTATION**: The following language and notation should be used throughout the course.

(1) Line Representation

$\overline{AB}$ means the line segment with endpoints A and B.

AB means the length or measure of $\overline{AB}$.

$\overleftrightarrow{AB}$ means the line determined by points A and B.

$\overrightarrow{AB}$ means a ray with endpoint A and passing through point B.

$\overline{ABC}$ means $\overline{AC}$ with point B on the segment and between A and C.

$\overline{ABCD}$ means $\overline{AD}$ with points B and C on the segment in the indicated order, that is, with B between A and C, and C between B and D. An extension of n ordered collinear points may also be used.

(2) Angle Measure

m $\angle$ ABC means the measure of $\angle$ ABC in degrees, written m $\angle$ ABC = 30.

When "the measure of" is written out, the degree symbol is included, i.e., the measure of $\angle$ ABC is 30°.

(3) Congruence

The symbol $\cong$ should be used to designate line segments of equal length, angles of equal measure, and congruent polygons.

**AN APPROACH**: It is hoped that this program will give the students a useful and enjoyable experience with mathematics which will encourage them to continue mathematics studies. It is suggested that the teacher give priority to a classroom atmosphere where there is ample time for student investigation, discovery, and discussion rather than to the task of "covering the material." Given sufficient time for the various possibilities these topics have to offer, the students will participate in their high school mathematics with a fuller appreciation of the fact that mathematics is a broader and more exciting field of study than many students perceive it to be.

It is our belief that the program outlined in this mathematics *Syllabus/Guide* reflects what is happening in mathematics today. It is our hope that the students who participate in it will be more likely to accept mathematics as an agreeable and relevant part of their total experience and want to continue with it, for their own benefit, and for the benefit of society.

# Goals and Learner Outcomes

The Statement of Regent's Goals for Elementary and Secondary Students—1984 includes the following:

- Each student will master computation skills as a foundation to:
  - —think logically and creatively
  - —apply reasoning skills to issues and problems
  - —perform basic mathematical calculations
  - —determine what information is needed for particular purposes and be able to acquire, organize, and use that information for those purposes.
- Each student will learn methods of inquiry and knowledge gained through mathematics, and use these methods and knowledge in interdisciplinary applications.

These Regents Goals act as a cornerstone for the State K-6 and 7-8 mathematics syllabi. The Course I and II syllabi build upon and extend this foundation.

The College Entrance Examination Board in "Academic Preparation for College" states that all college entrants will need to achieve proficiency in the following mathematical skills:

- the ability to apply mathematical techniques in the solution of real-life problems and to recognize when to apply those techniques
- familiarity with the language, notation, and deductive nature of mathematics and the ability to express quantitative ideas with precision
- the ability to use computers and calculators
- familiarity with the basic concepts of statistics and statistical reasoning
- knowledge in considerable depth and detail of algebra, geometry, and functions.

It is the goal of the syllabi for Courses I, II, III to provide students with a mathematical background which will be sufficient to meet their needs now and as they move into the 21st century.

The cultural aspects of mathematics should be made evident to learners so they can look upon it as a human invention important in itself—perhaps the greatest intellectual invention of the human mind. It has logic, abstraction, paradoxes, and patterns. These attributes should not be overlooked in mathematics instruction.

Mathematics can also be remarkably useful for dealing with real problems. Such problems should be an integral part of the students' high school mathematics program. Additionally, effective mathematics programs should help the students to develop an appreciation of mathematical ideas and thoughts, and should cultivate a trust and belief in results based upon mathematics. They should provide for the:

- mathematical competence necessary to function and live independently in society
- mathematical competence in areas essential to function in the work an individual does or aspires to do
- mathematical competence needed to engage in hobbies, games, and leisure time pursuits and add enjoyment to life.

Mathematics should benefit the students in three major goal areas:

- affective goals—including the appreciation of the structure of mathematics as a liberal art, and an understanding of the role that mathematics has played as a part of our cultural heritage
- content goals—including knowledge, skills, and the language of mathematics
- process goals—including interpretation, application, deduction, and transfer of mathematical concepts and processes.

General cognitive learner outcomes for Course II include:

- an understanding of the Laws of Reasoning, providing the students with formal deductive thinking skills through formal "logic" proofs

- a knowledge of the properties of the real number system which govern the operations on algebraic expressions
- the mastery of algebraic skills for its unifying value in mathematical thinking and for its application in practical formulas that appear in other disciplines, especially in the natural and social sciences
- the use of the axiomatic system of Euclidean geometry, with transformations and analytics, to prove geometric concepts and relationships
- extending the ability to solve quadratic equations, and developing a deeper understanding of quadratic relations by studying the graphic representations of the conic sections
- an understanding of combinatorics as an aid in the analysis and investigation of some special situations and in the solution of problems involving probability concepts
- the development of problem-solving strategies to solve a wide range of problems involving mathematical applications.

Specific content for Course II is listed in the Course II Outline (pages 11 to 14) of this *Syllabus/Guide* and is described more fully in the sections that follow the Course Outline. Instruction should be designed to integrate the topics as much as possible, emphasizing the relations between the various segments of the curriculum. Problem solving should be infused throughout the course of study, with concepts and techniques from the various strands introduced as they contribute to an understanding and solution of a problem.

# The Role of Problem Solving

Problem solving should be an integral part of the learning activities for each unit. It should be a classroom activity that goes beyond the topics of the syllabus to stimulate discussion, to encourage creativity, and to build a classroom atmosphere in which the students are encouraged to trust their abilities to think and to confront an unfamiliar task.

Teachers are encouraged to familiarize themselves with the problem-solving activities of the 7th and 8th grade syllabi and of the Course I syllabus. As the material of this course is developed in the classroom, the strategies with which the students are already familiar can be utilized as a foundation.

The model below shows the steps in solving a problem.

PROBLEM-SOLVING MODEL:

1. Read and understand the problem.
2. Organize the information—diagram, lists, etc.
3. Devise a plan for solving the problem.
4. Where appropriate, estimate the answer.
5. Carry out the plan.
6. Check the solution.

The study of logic presents a unique type of problem-solving situation. In Course I, the students had some experience in drawing conclusions based on logic. The rules of inference developed in Course II should extend and formalize their ability to solve logic problems. In addition to the type of logic proof suggested in this syllabus, the teacher may find the puzzle type of logic problems a valuable stimulus to discussion and creative thinking.

**EXAMPLE:**

Ann, Beth, Carol, Dave, and Ernie have birthdays on consecutive days, but not in the order in which their names are listed.

Ann's birthday is as many days before Carol's as Beth's is after Ernie's.

Dave is two days older than Ernie.

Carol's birthday is on a Wednesday this year.

On what days of the week are the birthdays of the others this year?

**Solution:**

Sunday—Dave
Monday—Ann
Tuesday—Ernie
Wednesday—Carol
Thursday—Beth

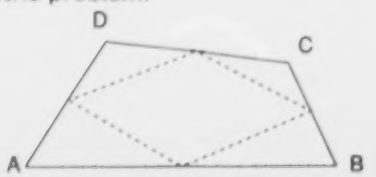
Geometric proof is a familiar type of problem-solving activity in mathematics. In Course I, the proof was intuitive rather than formal. In Course II, the students are expected to be able to write formal proofs of original geometric relationships.

**EXAMPLE:**

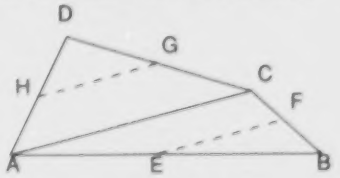
Prove that the figure formed by joining the consecutive midpoints of the sides of any quadrilateral is a parallelogram.

**Solution:**

A diagram is a useful aid in the solution of a geometric problem.




A method by which this statement can be proved may not readily occur to the students. In a classroom discussion, the teacher may find it useful to relate the problem to one using triangles by drawing a diagonal $\overline{AC}$ and the line segments joining the midpoints of $\overline{AB}$ and $\overline{BC}$ and the midpoints of $\overline{AD}$ and $\overline{DC}$.



$EF = \frac{1}{2} AC$

$HG = \frac{1}{2} AC$

Many problems lend themselves to a variety of solutions, and teachers may wish to return to a problem at different times during the course to encourage students to develop a different method of solution.

**EXAMPLE:**

Prove that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices.

**Solutions:**

One possible proof of this statement uses coordinates (See the section on Analytic Geometry).

A second possible method of solution uses an indirect proof.

Since D is the midpoint of $\overline{AB}$, AD = DB

If AD ≠ CD, then AD > CD or AD < CD

Assume AD > CD

$m\angle ACD > m\angle A$

DB > CD

$m\angle BCD > m\angle B$

$m\angle ACD + m\angle BCD > m\angle A + m\angle B$

But $m\angle ACD + m\angle BCD = 90$

and $m\angle A + m\angle B = 90$

Therefore we have a contradiction.

In a similar way it can be shown that a contradiction will arise if AD < CD.

Therefore we must conclude that AD ≠ CD is false or AD = CD is true.

An algebraic solution was an important strategy learned in Course I that should be reviewed and extended. As the students develop new algebraic skills, new problem-solving challenges are possible.

Problems that require the use of factors and products can provide valuable practice.

**EXAMPLE:**

Prove that the square of any prime number greater than 3 leaves a remainder of 1 when divided by 12.

**Solution:**

Since any prime number greater than 3 is odd, it can be written as 2n + 1.

$(2n + 1)^2 = 4n^2 + 4n + 1$

There is a remainder of 1 when divided by 4.

Since any prime number greater than 3 is not a multiple of 3, it can be written as 3n + 1 or 3n + 2.

$(3n + 1)^2 = 9n^2 + 6n + 1$

$(3n + 2)^2 = 9n^2 + 12n + 4$

$= 9n^2 + 12n + 3 + 1$

Either of these has a remainder of 1 when divided by 3.

Therefore, since 3 and 4 have no factor in common, the square of any prime number greater than 3 has a remainder of 1 when divided by (3)(4) or 12.

Many problems involving a geometric theorem can be proved using algebra.

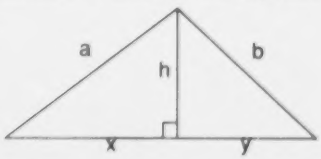
**EXAMPLE:**

Let a and b represent the measures of the legs of a right triangle and h the measure of the altitude to the hypotenuse of the right triangle. Prove that:

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{h^2}$$

**Solution:**

Let the measures of the segments into which the hypotenuse is divided be represented by x and y.




$a^2 = x(x + y)$ $b^2 = y(x + y)$ $h^2 = xy$

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{x(x + y)} + \frac{1}{y(x + y)}$$

$$= \frac{y}{xy(x + y)} + \frac{x}{xy(x + y)}$$

$$= \frac{y + x}{xy(x + y)}$$

$$= \frac{1}{xy}$$

$$= \frac{1}{h^2}$$

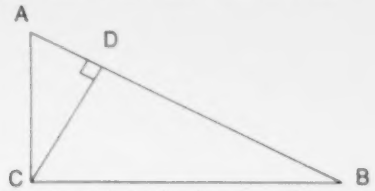
Many applications use a combination of algebra and geometry for solution.

**EXAMPLE:**

City A is 30 miles directly north of city C, and city B is 40 miles directly east of city C. Find

the length of the shortest path from city C to a straight road from city A to city B.

**Solution:**




The shortest path is the length of the perpendicular from C to $\overline{AB}$, CD. To find CD, first find AB, AD, and BD.

It is also possible to use the result of the previous problem.

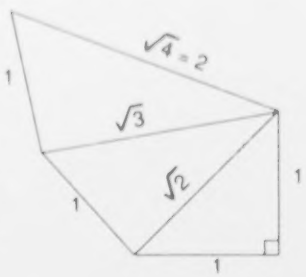
Constructions offer a valuable source of material for original problem solving.

**EXAMPLE 1:**

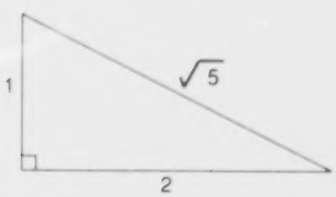
Construct a segment of length $\sqrt{5}$ if a segment of length 1 is given.

**Solution:**

Construct an isosceles right triangle whose legs have measure 1. Using the hypotenuse as one leg, construct a right triangle whose other leg has measure one. Therefore the hypotenuse of this triangle has measure $\sqrt{3}$. By repeating this last construction, it is possible to construct a line segment whose length is the square root of each positive integer.




A shorter construction uses the equation $1^2 + 2^2 = 5$. Therefore the required line segment is the hypotenuse of a right triangle whose legs have measures 1 and 2.




**EXAMPLE 2:**

Construct a "Golden Rectangle."

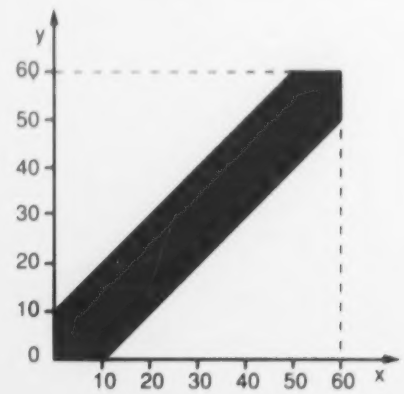
Geometry and probability can provide the solutions to some challenging problems.

**EXAMPLE:**

A boy and a girl agree to meet under the big clock at the railroad station. They agree that each will arrive sometime between five and six o'clock, wait no more than ten minutes, and, if the other has not arrived in that time, leave. What is the probability that, if they follow this plan, the boy and girl will, in fact, meet each other?

**Solution:**

Let x represent the number of minutes after five that the boy arrives and y represent the number of minutes after five that the girl arrives ($0 < x < 60$ and $0 < y < 60$). If $x-y < 10$, the boy and the girl will meet.




The ratio of the area of the shaded region to the entire region is the required probability, $\frac{11}{36}$.

Often the solution of a problem can be extended to gain new insights about the topic.

**EXAMPLE:**

Find two consecutive positive integers whose squares differ by 75.

**Solution:**

Let $x$ = the smaller integer
$x+1$ = the larger integer

$$\begin{aligned} (x+1)^2 - x^2 &= 75 \\ (x^2 + 2x + 1) - x^2 &= 75 \\ 2x + 1 &= 75 \\ 2x &= 74 \\ x &= 37 \end{aligned}$$

$38^2 = 1444$ and $37^2 = 1369$

$1444 - 1369 = 75$

Therefore the consecutive integers are 37 and 38. This problem can lead to new understanding of consecutive integers and their squares through questions such as the following:

In the solution of the equation, the difference of the squares of the two consecutive integers is $2x + 1$ or $x + (x + 1)$, that is, the sum of the integers. Therefore,

$38^2 - 37^2 = 37 + 38$

or $38^2 = 37^2 + 37 + 38$

Can the difference between the squares of two consecutive integers be even?

If it is known that the square of 46 is 2116, how can addition be used to find the square of 47?

What is the difference between the squares of two consecutive even integers or two consecutive odd integers?

Can these differences be written in terms of the integers?

How is the problem changed if the integers need not be positive?

The ability to confront non routine tasks, that is, to deal successfully with those tasks that require the selection of a procedure that is not immediately obvious, is a desirable outcome of education in mathematics. This skill does not develop automatically with the mastery of computational skills and techniques, but must be developed through practice, through the sharing of insights, and through the development of the habits of questioning and curiosity.

As teachers of mathematics, we see it as a logical, coherent set of concepts that develop naturally one from another. Many of our students do not see it that way because, often, in the interest of "covering the course" or "completing the work" of a particular year, mathematics has been presented as a set of rules to be learned rather than as ideas that unfold or emerge from the learner's ability to see and appreciate relationships. Time given to problem solving can be time well spent because the enthusiasm and excitement which can be generated lead to more retentive learning.

The purpose, then, of this introductory section is to encourage the use of non routine problems throughout the course in addition to those problems that are part of the content of each unit of the course. When chosen appropriately, classroom activities and applications can serve a multi-purpose role in the applying of mathematics concepts and skills to important issues. For example, awareness of various handicapping conditions can be explored by applying such concepts and skills to the use of wheelchair ramps, medication and/or adaptive devices, etc.

Problems may serve the following purposes:

1. to motivate the introduction of a new topic or the development of a new skill or technique,
2. to provide practice with a skill that has been or is being developed,
3. to allow the exploration of concepts beyond those suggested by the content of the course, and
4. to develop in the individual student the confidence to try various techniques when faced with an unfamiliar problem.

# Course II Outline


I. Logic ..... 15

A. Review of Logic from Course I ..... 15

1. Logical connectives and truth values
2. Related conditionals
   a. converse
   b. inverse
   c. contrapositive

B. Laws of Reasoning ..... 16

1. Law of Detachment
2. Law of Contrapositive
3. Law of Disjunctive Inference
4. Law of Syllogism
5. DeMorgan's Laws

C. Development of Notions of Proof ..... 20

1. Direct
2. Indirect

II. Mathematical Structure and Related Algebraic Skills ..... 23

A. Review of Properties of the Real Numbers ..... 23

1. Closure
2. Commutative property
3. Associative property
4. Identity element
5. Inverse element
6. Distributive property
7. The importance of the properties of the real numbers

B. Review of Algebraic Skills ..... 26

1. Addition of polynomials
2. Multiplication of polynomials
3. Simplification of algebraic expressions
4. Division of polynomials by monomials
5. Factoring
6. Solution of linear equations

C. Extension of Algebraic Skills ..... 29

1. Factoring
2. Simplifying algebraic fractions with polynomial denominators
3. Multiplication and division of fractions with polynomial denominators
4. Addition and subtraction of fractions

D. Groups and Fields (Optional) ..... 31
1. Groups
a. definition
b. infinite systems
c. finite systems
d. theorems
2. Fields
a. definition
b. applications

III. Euclidean Geometry ..... 35
A. Axiomatic Systems ..... 35
1. Undefined terms and definitions
2. Assumptions
3. Theorems and proof

B. Congruence ..... 36
1. Assumptions
2. Isosceles triangle theorems
3. Other congruence theorems (AAS, HL)

C. Inequalities ..... 37
1. Angle relationships
2. Relationships of lengths of sides

D. Parallelism ..... 37
1. Definitions and assumptions
2. Theorems
3. Angle measure in triangles
4. Classification and properties in quadrilaterals

E. Similarity ..... 39
1. Definition of similar polygons
2. Similar triangles
3. Pythagorean relationship
4. Special right triangles
5. Right triangle trigonometry

F. Constructions and Proofs ..... 41
1. Basic constructions
2. Other constructions

IV. Analytic Geometry ..... 45
A. Review of the Rectangular Coordinate System ..... 45
B. Parallel and Perpendicular Lines ..... 45
1. Parallel lines
2. Perpendicular lines

C. Formulas ..... 46
1. Length of a segment
2. Midpoint of a segment

D. Equation of a Line ..... 47
1. Point-slope form
2. Slope-intercept form

E. Area ..... 48
1. Review of formulas
2. Application to coordinate geometry
F. Transformations ..... 49
1. Reflection in a line
2. Reflection in a point
3. Translations
4. Dilations
G. Conics ..... 51
1. Circle
2. Parabola
3. Other conics
4. Graphic solution of systems of equations
H. Locus ..... 53
1. Definition
2. Basic loci
a. At a fixed distance from a point
b. At a fixed distance from a line
c. Equidistant from two points
d. Equidistant from two parallel lines
e. Equidistant from two intersecting lines
3. Compound locus
4. A special locus
I. Analytic Proof ..... 54
V. Equations and Inequalities ..... 57
A. Review ..... 57
1. Linear equations and inequalities
2. Proportions
B. Fractional Equations ..... 59
C. Quadratic Equations ..... 59
1. Solution by factoring
2. Solution by completing the square
3. Solution by formula
4. Applications
D. Pairs of Equations ..... 62
1. Review of pairs of linear equations
2. Quadratic-linear pair
VI. Probability and Combinatorics ..... 65
A. Review of Fundamentals ..... 65
1. Probability concepts
2. Permutation concepts
a. n objects, all different, taken n at a time
b. n objects, all different, taken r at a time, $r \leq n$
B. Extension of Work with Permutations ..... 66
1. Number of permutations of n objects, r identical, taken all at a time
2. Number of permutations of n objects, r identical, s identical, t identical, etc., taken all at a time

C. Combinations ..... 67

1. Distinction between combination and permutation
2. Combinations of n objects, taken r at a time, $r \leq n$
3. Applications

**A Note on Time and Sequence**

It is difficult to supply a meaningful time frame for an integrated course since it is intended that topics interact with each other in the development of the material. The following chart should be used only as a general frame of reference.

| *Topic* | *Days* |
|---|---|
| I. Logic | 10 - 15 |
| II. Mathematical Structure and Related Algebraic Skills | 10 - 15 |
| III. Euclidean Geometry | 60 - 65 |
| IV. Analytic Geometry | 30 - 35 |
| V. Equations and Inequalities | 15 - 20 |
| VI. Probability and Combinatorics | 10 - 15 |
| TOTAL | 135-165 |

Depending on the students' backgrounds, time allotments for one or more of these topics may vary significantly.

The above chart is not intended to imply that the topics should be taught in that order or as separate blocks of material.

# I.

# *Logic*

In Course I, the study of Logic was designed to give the students a background in analyzing given information, translating written statements into symbolic form, and drawing sound conclusions. The emphasis was on the logical connectives, the truth values of compound statements using these connectives, and the use of truth tables to prove tautologies.

The Logic unit in Course II develops an understanding of the Laws of Reasoning and emphasizes more formal deductive thinking skills through an introduction to mathematical proof.

**REVIEW OF LOGIC FROM COURSE I**

### *Logical Connectives and Truth Values*

The connectives AND ($\wedge$), OR ($\vee$), IF . . . THEN ($\rightarrow$), and IF AND ONLY IF ($\leftrightarrow$) have been studied in Course I. The students should review translating statements into symbolic form and writing negations of simple statements.

The basic truth values for the logical connectives are summarized here:

| p | q | $p \wedge q$ | $p \vee q$ | $p \rightarrow q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | F | T | T | F |
| F | F | F | F | T | T |

The students should demonstrate their ability to use these rules for determining the truth values of compound statements and in proving tautologies.

**EXAMPLE 1:**

Determine the truth value of each statement:

a) The LCM of 6 and 9 is 18 and the GCF of 12 and 48 is 6.

b) If 15 is not divisible by 2, then 15 is prime.

c) The product of two negative numbers is negative or $-8 < -2$.

**EXAMPLE 2:**

Determine whether or not the following statement is a tautology:

$\sim (p \rightarrow q) \leftrightarrow (\sim p \vee q)$

### *Related Conditionals*

The students should review the three conditionals related to the conditional $(p \rightarrow q)$.

| | |
|---|---|
| Original | $p \rightarrow q$ |
| Converse | $q \rightarrow p$ |
| Inverse | $\sim p \rightarrow \sim q$ |
| Contrapositive | $\sim q \rightarrow \sim p$ |

Examples should be given illustrating that neither the converse nor the inverse necessarily has the same truth value as the original, but that the contrapositive will always have the same truth value as the original.

**EXAMPLE:**

| | |
|---|---|
| Original: | If a figure is a square, then it is a parallelogram. TRUE |
| Converse: | If a figure is a parallelogram, then it is a square. FALSE |
| Inverse: | If a figure is not a square, then it is not a parallelogram. FALSE |
| Contrapositive: | If a figure is not a parallelogram, then it is not a square. TRUE |

## LAWS OF REASONING

In Course I, the Logic unit developed the foundations for several important patterns of deductive thinking. In Course II, these patterns of reasoning (laws of inference) are now formalized, and the students are expected to recognize and use these laws in drawing sound conclusions.

### *Law of Detachment*

Perhaps the simplest and most straightforward rule, the Law of Detachment, states that if a conditional is true and its antecedent is true, then its consequent is true. Symbolically, this is stated:

$[(p \rightarrow q) \wedge p] \rightarrow q$

An informal explanation of the Law of Detachment can be found by studying the truth table for the conditional:

| p | q | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

The only line in which both the conditional ($p \rightarrow q$) and its antecedent (p) are true is the first. In this line the consequent (q) is also true.

A more formal demonstration of the validity of this pattern of reasoning is the construction of a truth table to show that $[(p \rightarrow q) \wedge p] \rightarrow q$ is a tautology.

| p | q | $p \rightarrow q$ | $(p \rightarrow q) \wedge p$ | $[(p \rightarrow q) \wedge p] \rightarrow q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | T |

After understanding the form of the Law of Detachment pattern of reasoning, the students should work with problems stated either in words or in symbols. Examples such as the following might be used.

**EXAMPLE 1:**

If today is Wednesday, then yesterday was Tuesday. Today is Wednesday.

**Conclusion:**

Yesterday was Tuesday.

**EXAMPLE 2:**

John is restricted to a wheelchair. If a student needs a wheelchair, he will need access ramps.

**Conclusion:**

John will need access ramps

**EXAMPLE 3:**

Peter received an A in French. If a student receives an A in a course, he is exempt from the final exam.

**Conclusion:**

Peter is exempt from the French final exam.

**EXAMPLE 4:**

If population increases, then man's needs for natural resources increase. Population is increasing.

**Conclusion:**

Our needs for natural resources are increasing.

**EXAMPLE 5:**

If $0 < a < 1$, then $a^2 < a$. $a = \frac{1}{2}$.

**Conclusion:**

$$\left(\frac{1}{2}\right)^2 < \frac{1}{2}$$

$$\frac{1}{4} < \frac{1}{2}$$

**EXAMPLE 6:**

$\sim p$. $\sim p \rightarrow q$

**Conclusion:**

q

***Law of Contrapositive***

This law states an idea from Course I with which the students are familiar—A conditional statement and its contrapositive always have the same truth value.

Symbolically:

$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

The validity of the law can be shown by a truth table demonstrating that it is a tautology.

| p | q | $p \rightarrow q$ | $\sim q$ | $\sim p$ | $\sim q \rightarrow \sim p$ | $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

A consequence of the Law of Contrapositive and the Law of Detachment is the Law of Modus Tollens, which states that if a conditional is true and its consequent is false, its antecedent must also be false.

$[(p \rightarrow q) \wedge \sim q)] \rightarrow \sim p$

This can be demonstrated by examining the truth table for the conditional and finding that the fourth line is the only line where the conditional is true and its consequent is false. In this line the antecedent is also false.

| p | q | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

It can also be shown that this law can be derived by using the Law of Detachment with the contrapositive of the conditional. Since the contrapositive and the conditional have the same truth value (are logically equivalent),

$[(\sim q \rightarrow \sim p) \wedge \sim q)] \rightarrow \sim p$

means that $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$.

Examples such as the following illustrate the use of these laws:

**EXAMPLE 1:**

Write a logically equivalent conditional for each of these statements:

a) $(r \vee \sim s) \rightarrow t$

**Answer:**

$\sim t \rightarrow \sim (r \vee \sim s)$

b) If a quadrilateral is a rectangle, then it is a parallelogram.

**Answer:**

If a quadrilateral is not a parallelogram, then it is not a rectangle.

**EXAMPLE 2:**

Use the Law of Contrapositive to draw a conclusion:

a) If a year is a leap year, then February has 29 days. February does not contain 29 days.

**Conclusion:**

This is not a leap year.

b) $\sim p \rightarrow r$
$\sim r$

**Conclusion:**

$\sim (\sim p)$ or $p$

***Law of Disjunctive Inference***

This law states that if a disjunction $(p \vee q)$ is true and one of its parts is false, then the other part must be true. The students can see the reasonableness of this by inspecting the basic truth table. There are two lines which meet the conditions. In either case, the other premise is true.

| p | q | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Another way to verify this law is to construct a truth table to show that both of the following are tautologies:

$[(p \vee q) \wedge \sim q] \rightarrow p$

$[(p \vee q) \wedge \sim p] \rightarrow q$

Examples should be given to the students in both symbolic and word forms. Using the reasoning pattern of Disjunctive Inference, the students should be able to draw conclusions for problems such as:

**EXAMPLE 1:**

$a = 0$ or $b = 0$
$b \neq 0$

**Conclusion:**

$a = 0$

**EXAMPLE 2:**

A positive integer greater than 1 is either a prime or a composite number.
15 is not a prime number.

**Conclusion:**

15 is a composite number.

**EXAMPLE 3:**

$(r \vee \sim s) \wedge s$

**Conclusion:**

r

It should be noted that $s \leftrightarrow \sim (\sim s)$ in the foregoing example. This is sometimes known as the Law of the Double Negative and can be related to the students' work with signed numbers in arithmetic $(-(-4) = 4)$.

***Law of Syllogism***

This law of reasoning uses the conjunction of the two conditionals $(p \rightarrow q)$ and $(q \rightarrow r)$ to produce a third conditional $(p \rightarrow r)$. An analogy can be made between this law $[(p \rightarrow q) \wedge (q \rightarrow r)] \leftrightarrow (p \rightarrow r)$ and the transitive properties of equality or inequality (If $a = b$ and $b = c$, then $a = c$, or if $a < b$ and $b < c$, then $a < c$).

A truth table to prove the Syllogism Law as a tautology requires eight rows since there are three variables.

The Law of Syllogism is very commonly used in mathematics. In the solution of an equation, for example, we reason as follows:

If $3x+5 = 17$, then $3x = 12$
If $3x = 12$, then $x = 4$
Therefore, if $3x+5 = 17$, $x = 4$.

Examples such as the following may be considered:

**EXAMPLE 1:**

If Mary passes three years of math, she has completed a sequence. If she completes a sequence, she may take electives.

**Conclusion:**

If Mary passes three years of math, she may take electives.

**EXAMPLE 2:**

$t \rightarrow (s \wedge w)$
$(s \wedge w) \rightarrow k$

**Conclusion:**

$t \rightarrow k$

The Law of Syllogism can be effectively used to illustrate the difference between the validity of a logical reasoning process and the truth of the conclusions.

In the following example, a true conclusion is drawn from false premises through a valid reasoning process.

If a figure is a square, then it is a trapezoid.
If a figure is a trapezoid, then it is a parallelogram.
Therefore, if a figure is a square, then it is a parallelogram.

In the same manner, a false conclusion could be derived from false premises, even though the reasoning process was valid. The students should note the importance of ascertaining the truth of premises as well as the importance of using correct reasoning processes in order to arrive at both correct and valid conclusions.

***DeMorgan's Laws***

DeMorgan's Laws, which were developed in set theory, may be applied to Logic. The two DeMorgan Laws give rules for negating conjunctions and disjunctions. These laws are symbolically stated as follows:

$\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
$\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$

It should be noted that the negation of a conjunction is a disjunction and that the negation of a disjunction is a conjunction. These laws can easily be verified using a truth table to demonstrate that each is a tautology. The following truth table shows the first DeMorgan Law to be a tautology.

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \wedge q$ | $\sim (p \wedge q)$ | $\sim p \vee \sim q$ | $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | F | T |
| T | F | F | T | F | T | T | T |
| F | T | T | F | F | T | T | T |
| F | F | T | T | F | T | T | T |

Using this DeMorgan Law, we can write the negation of the conjunction, "Mary passed French and Sally passed Latin." This statement would be false if either girl failed her subject *or* if both failed their subjects. Thus the negation of "Mary passed French and Sally passed Latin" is "Mary did *not* pass French *or* Sally did *not* pass Latin." Recall that *or* means *either* or *both*.

Using the second DeMorgan Law, we can write the negation of a disjunction such as "Today is Tuesday or Wednesday." We must assert that today is neither Tuesday nor Wednesday. That is, the negation is, "Today is *not* Tuesday *and* today is *not* Wednesday."

The students should apply these laws to statements in words or symbols.

**EXAMPLE 1:**

Negate "x is prime and $x > 41$."

**EXAMPLE 2:**

Write an equivalent statement for

$\sim (r \vee \sim s)$.

**DEVELOPMENT OF NOTIONS OF PROOF**

The notion of proof in mathematics is a very important one. Throughout the study of arithmetic, algebra, geometry, and other branches of mathematics, we frequently ask, "Are we certain this is always true?", or "Can you prove it?" The Laws of Reasoning studied in this Logic unit form the foundation for drawing sound

conclusions and organizing formal proofs. Several of the units in Course II emphasize proving ideas that were developed on an intuitive level in Course I. This section of the Logic unit discusses two types of proof which will be particularly useful in the Geometry units—direct proof and indirect proof.

***Direct***

A direct proof sequences the given premises in such a way that conclusions can be drawn using one or more of the Laws of Reasoning. This orderly arrangement of the steps in reaching the final conclusion can be written in paragraph form or in a two column form. In both cases, reasons should be given.

**EXAMPLE:**

Given: Either Fred or Sam murdered Mr. Green. If Sam murdered Mr. Green, then the butler was off Thursday night.

If the butler was off Thursday night, then Nurse Partridge did not witness the crime.

However, it is known that Nurse Partridge did witness the crime.

Determine: Who murdered Mr. Green?

*Prove* your result.

**Solution:**

1. Symbolize each premise:
   Let F represent "Fred murdered Mr. Green."
   Let S represent "Sam murdered Mr. Green."
   Let B represent "The butler was off Thursday night."
   Let P represent "Nurse Partridge witnessed the crime."
   Thus the original premises can be symbolized:

   $F \vee S$
   $S \cdot B$
   $B \cdot \sim P$
   $P$

2. Study the premises written symbolically and reorder them in such a way that logical conclusions can be drawn using the Laws of Inference.

3. Write the proof in two columns, giving the reasons for each statement in the second column. Reasons can be any of the Laws of Inference or the fact that the statement is part of the given information.

| | *Statements* | | *Reasons* |
|---|---|---|---|
| 1. | $S \cdot B$ | 1. | Given |
| 2. | $B \cdot \sim P$ | 2. | Given |
| 3. | $S \cdot \sim P$ | 3. | Law of Syllogism (1, 2) |
| 4. | $P$ | 4. | Given |
| 5. | $P \cdot \sim S$ | 5. | Law of Contrapositive (3) |
| 6. | $\sim S$ | 6. | Law of Detachment (4, 5) |
| 7. | $F \vee S$ | 7. | Given |
| 8. | $F$ | 8. | Law of Disjunctive Inference (6, 7) |

This sequence of steps and reasons establishes that Fred murdered Mr. Green. It should be noted that other sequences of steps and reasons could reach the same conclusion.

The numbers in parentheses after the Law of Inference show which statements are being used with that particular law.

The basic notions of a direct proof—sequencing the premises, drawing conclusions which can be used with other premises or conclusions to reach

a final conclusion, and supporting these conclusions with an accepted reason—are the same whether the proof is a logic proof, an algebraic proof, a geometric proof, or an analytic proof. Working with proofs in symbolic logic allows the students to use the familiar Laws of Inference to develop the concept of proof before they are expected to write proofs in algebra or geometry.

***Indirect***

The method of indirect proof is a powerful mathematical tool. Rather than prove directly that the premises lead to a conclusion, the conclusion is assumed to be false and then by a series of logically sound reasoning steps, it is shown that this assumption leads to a contradiction of one of the premises or some other known fact. Reaching such a contradiction or false statement means that the assumption was incorrect. Thus the conclusion is not false; it is true.

Using the same example as in the direct proof section, the following is an indirect proof demonstration that "Fred murdered Mr. Green." This proof is written in two column form, but some teachers may wish to use a paragraph form for indirect proofs. The form the proof takes is arbitrary—in either case, the essence of the indirect proof is the same.

| *Steps* | *Reasons* |
|---|---|
| 1. $\sim F$ | 1. Assumption |
| 2. $F \vee S$ | 2. Given |
| 3. $S$ | 3. Law of Disjunctive Inference (1, 2) |
| 4. $S \cdot B$ | 4. Given |
| 5. $B$ | 5. Law of Detachment (3, 4) |
| 6. $B \cdot \sim P$ | 6. Given |
| 7. $\sim P$ | 7. Law of Detachment (5, 6) |
| 8. $P$ | 8. Given |
| 9. $P \wedge \sim P$ is false; that is, a contradiction has been reached. | 9. Definition of conjunction (7, 8) |
| 10. $\sim (\sim F)$ | 10. If an assumption leads to a contradiction, it must be false. |
| 11. $\therefore F$ | 11. Law of the Double Negative |

The basic idea used in the indirect proof—assuming the desired conclusion false and showing that this assumption leads to a contradiction—is closely related to the Contrapositive Law. Rather than prove $p \rightarrow q$ directly, $\sim q \rightarrow \sim p$ is proven and thus its contrapositive, $p \rightarrow q$, must also be true.

Further work with indirect proof will be found in later units of this course in sections dealing with parallel lines and with inequality relationships in triangles.

# II.

# *Mathematical Structure and Related Algebraic Skills*

The properties of the real number system govern the operations on algebraic expressions. The first section of this unit reviews the properties of the real numbers which should be discussed and used in the development of algebraic skills. An optional topic in this unit allows for a more formal study of the structure of groups and fields.

## REVIEW OF PROPERTIES OF THE REAL NUMBERS

### *Closure*

A set S is closed under a binary operation, $*$, if for all a and b in S, $a * b$ is an element of set S.

The set of real numbers, as well as many of its subsets, e.g., the rationals, integers, and whole numbers, is closed with respect to addition and multiplication. The rational numbers are closed with respect to subtraction and division (excluding 0), but the whole numbers are not.

### *Commutative Property*

An operation $*$ is commutative on a set S if for all a and b in S, $a * b = b * a$.

Commutativity is the principle behind reordering the terms in an arithmetic or algebraic expression, e.g., $3x + 2 = 2 + 3x$ because addition is commutative on the real numbers. The reals, rationals, and integers are commutative with respect to addition and multiplication. None of these sets is commutative with respect to subtraction or division.

### *Associative Property*

An operation $*$ is associative on a set S if for all a, b, and c in S,
$(a * b) * c = a * (b * c)$.

Associativity is the principle behind regrouping of terms. The equation $3x + (2x + 5y) = (3x + 2x) + 5y$ is an example of the associativity of addition on the reals. The reals and rationals are associative with respect to addition and multiplication, but are not associative with respect to subtraction and division (zero excluded).

Operations other than the four basic arithmetic operations should be used to reinforce the understanding of these properties.

**EXAMPLE 1:**

Let $*$ represent finding the average or mean of two numbers, i.e.

$$a * b = \frac{a + b}{2}$$

a. Show that the integers are not closed with respect to this operation.

b. Prove that $*$ is commutative on the reals.

$$a * b = \frac{a + b}{2} = \frac{b + a}{2}$$

$$\frac{b + a}{2} = b * a$$

$$a * b = b * a$$

Since $a * b = b * a$ for any choice of a and b, the desired conclusion has been demonstrated.

c. Show that $*$ is not associative on the reals and the rationals. Virtually any example will show this, e.g.

$$(4*2)*8 = 3*8 = \frac{11}{2} = 5.5$$

$$4 * (2*8) = 4 * 5 = \frac{9}{2} = 4.5$$

$$5.5 \neq 4.5$$

**EXAMPLE 2:**

Let $\Delta$ represent an exponential relation, i.e. $a \Delta b = a^b$

a. Show that the whole numbers are closed with respect to this operation but that the integers are not closed.

b. Show that $a \Delta b \neq b \Delta a$.

c. Show that $\Delta$ is not associative on the reals.

***Identity Element***

If $*$ is a binary operation defined on set S, then e is an identity element for the system if for all a in S, $a * e = e * a = a$. The number 0 is the additive identity for the set of real numbers and its usual subsets; for any r in R, $r + 0 = 0 + r = r$. The number 1 is the multiplicative identity for the reals and its usual subsets; for any r in R, $r \cdot 1 = 1 \cdot r = r$.

**EXAMPLE:**

Show that 0 is the identity for the set of real numbers under the operation $a*b = a+b-ab$.

***Inverse Element***

In a system with identity e, b is the inverse of a if $a * b = b * a = e$.

The usual notation is the familiar dash, or negative sign, to indicate an inverse in an additive system. Examples:

$4 + (-4) = 0$ in the reals

$-1.09 + (-(-1.09)) = 0$ in the reals

Two notations are used for the multiplicative inverse, $\frac{1}{a}$ and $a^{-1}$. The multiplicative inverse of $\frac{1}{12}$, which can be written either as $\left(\frac{1}{12}\right)^{-1}$ or $\frac{1}{\frac{1}{12}}$, is 12 since $\frac{1}{12} \cdot 12 = 1$.

Note: $\frac{1}{0}$ or $0^{-1}$, does not exist in the reals. Division by zero is not possible.

### *Distributive Property*

In a system with two operations, $\oplus$ and $\otimes$, $\otimes$ is said to distribute over $\oplus$ if, for all a, b, and c , $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.

The importance of the distributive property cannot be over-emphasized, for this is the property, or rule, which links the two operations of a system. In the set of reals and its usual subsets, multiplication distributes over addition, permitting the use of these two operations in the same problem, e.g. $5(3 + 30) = 5(3) + 5(30)$.

**EXAMPLE 1:**

Does multiplication distribute over subtraction in the reals? An example such as $10(7 - 2) = 10(7) - 10(2)$ leads to the conclusion that the answer to this question is yes. Of course, examples do not constitute a proof, but well-chosen examples can lead to good conjectures which may be proven.

**EXAMPLE 2:**

Does addition distribute over multiplication? The answer is no, as the following example shows.

$5 + (3 \cdot 7) \neq (5 + 3) \cdot (5 + 7)$

Using a table of a finite number of elements and an operation such as the following provides practice and aids in the understanding of the properties. The set T = {2, 4, 6, 8}, together with the operation * defined such that the product of any two elements is the units digit of the product under multiplication as shown in the table.

| * | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| 2 | 4 | 8 | 2 | 6 |
| 4 | 8 | 6 | 4 | 2 |
| 6 | 2 | 4 | 6 | 8 |
| 8 | 6 | 2 | 8 | 4 |

Using this table, the students should be able to illustrate the properties from the table. The table is read from the elements on the left through the operation to the elements on the top.

Closure: $2 * 4 = 8, 6 * 6 = 6, 8 * 4 = 2$, etc.
Commutativity: $4 * 8 = 2 = 8 * 4$, etc.
Associativity: $(2 * 4) * 6 = 8 * 6 = 8$
$2 * (4 * 6) = 2 * 4 = 8$, etc.
Identity Element: 6
Inverse: $2 * 8 = 6$; 8 is the inverse of 2, etc.

Looking at a system having elements other than numbers often helps the students to gain a greater understanding of the mathematical properties. The following system has closure, an identity element, and inverses, and is both commutative and associative. It can also be acted out by a student or the teacher, providing an interesting and motivating class activity.

| Ⓕ | S | L | R | A |
|---|---|---|---|---|
| S | S | L | R | A |
| L | L | A | S | R |
| R | R | S | A | L |
| A | A | R | L | S |

Ⓕ = followed by
S = Stand as you are
L = Left face
R = Right face
A = About face

i.e., R Ⓕ A = L

A right face followed by about face = left face.

### *The Importance of the Properties of the Real Numbers*

The properties presented in this unit lay the foundation for the definition of groups and fields given in the optional section of this unit. A thorough understanding of these properties provides the students with skills necessary in working with the algebra of polynomials.

Under the operation of addition, these properties can easily be verified for polynomials.

1) **Closure:** If you add two polynomials, you get a polynomial.

   Example: $(3x^2 - 2x + 1) + (x^4 + x - 3) = (x^4 + 3x^2 - x - 2)$

2) **Associativity:** Addition of polynomials is associative.

   Example: $[(2x^2 + 1) + (3x - 3)] + (x^3 \cdot 2x^2)$ is equivalent to $(2x^2 + 1) + [(3x - 3) + (x^3 - 2x^2)]$

3) **Additive Identity:** 0 is the additive identity for the set of polynomials.

   Example: $(3x^2 + 2x + 1) + 0 = 3x^2 + 2x + 1$

4) **Additive inverses:** Each polynomial has an additive inverse.

   Example: $(3x^2 + 2x + 1) + (-3x^2 - 2x - 1) = 0$

**The commutative property** also holds for the addition of polynomials.

## REVIEW OF ALGEBRAIC SKILLS

Algebraic expressions usually represent one or more real numbers. Thus the properties of the real numbers govern the operations on algebraic expressions. This section presents a summary and review of the algebraic techniques that the students should experience before moving on to the newer material of later sections.

### *Addition of Polynomials*

Adding polynomials, often referred to as combining like terms, is an operation based on the commutative, associative, distributive, identity, and inverse properties.

**EXAMPLE 1:**

Combine like terms: $3x + 4x + 5y$

The associative and distributive properties are used to obtain $7x + 5y$.

**EXAMPLE 2:**

Simplify: $(2x^2 + 3y + 3z^2) + (3x^2 + 4z^2 - 3y)$

In addition to being able to use the commutative, associative and distributive properties, the students need to recognize that $3y$ and $-3y$ are additive inverses and that their sum is the additive identity, 0.

### *Multiplication of Polynomials*

This algebraic skill is highly dependent upon an understanding of the distributive property of multiplication over addition.

**EXAMPLE 1:**

Multiply: $3z\left(8z - 5z^2 + \frac{1}{3}\right)$

**EXAMPLE 2:**

Multiply $(x + 5)(x + 8)$

The multiplication of two binomials requires repeated use of the distributive property. Similar terms are then combined.

$$(x + 5)(x + 8) = (x + 5)x + (x + 5)8$$
$$= x^2 + 5x + 8x + 40$$
$$= x^2 + 13x + 40$$

The following equality is a generalization of this example:

$(a + b)(c + d) = ac + ad + bc + bd$

A geometric interpretation of this rule, involving the area of a rectangle, appears at the right.

| | c | d |
|---|---|---|
| b | bc | bd |
| a | ac | ad |

The area of the large rectangle [(a+b)(c+d)] equals the sum of the areas of the smaller rectangles.

**EXAMPLE 3:**

Multiply: $(2x - y)(3x + 2xy + y^2)$

### *Simplification of Algebraic Expressions*

It is sometimes necessary to remove parentheses prior to combining like terms when simplifying algebraic expressions.

**EXAMPLE 1:**

Simplify: $2c + \frac{1}{2}(10c + 3)$

**EXAMPLE 2:**

Simplify: $6y - (2x - 9y)$

The students frequently have difficulty with problems of this type. The problem can be approached by the use of either the distributive property: $6y - (2x - 9y) = 6y + (-1)(2x - 9y)$, or the additive inverse property: $6y - (2x - 9y) = 6y + (-2x + 9y)$.

### *Division of Polynomials by Monomials*

In Course I, the students worked with those cases in which the denominator is a factor of the numerator.

**EXAMPLE 1:**

Divide: $\frac{16xy^3}{8y^2}$

The identity property is useful in this division example, which can also be considered as reducing the fraction.

Since $\frac{a}{a} \cdot b = 1 \cdot b = b$

Then, $\frac{16xy^3}{8y^2} = \frac{8y^2 \cdot 2xy}{8y^2} = \frac{8y^2}{8y^2} \cdot 2 \times y = 2xy$

**EXAMPLE 2:**

Divide: $\frac{3x^2 + 6xy}{3}$

Since division can be considered as multiplication by the reciprocal, this fraction can be rewritten as

$\frac{1}{3}(3x^2 + 6xy)$. Then the Distributive Law can be used:

$$\frac{1}{3}(3x^2 + 6xy) = \frac{1}{3}(3x^2) + \frac{1}{3}(6xy)$$
$$= x^2 + 2xy$$

An alternate approach uses the principle that division is distributive over addition, i.e.,

$$\frac{3x^2 + 6xy}{3} = \frac{3x^2}{3} + \frac{6xy}{3} = x^2 + 2xy$$

***Factoring***

From their work in Course I, the students should be familiar with the idea that factoring reverses multiplication. Basically then, factoring is an application of the distributive property. The students should be able to:

1. Remove the greatest common monomial factor (GCF)
2. Factor a trinomial into 2 binomials
3. Factor the difference of two squares

**EXAMPLES:**

$7x + 7y = 7(x + y)$

$30xy^2z - 18x^2yz = 6xyz(5y - 3x)$

$2x^2 - 11x + 12 = (2x - 3)(x - 4)$

$4x^2 - 1 = (2x + 1)(2x - 1)$

Factoring provides yet another approach to dividing a polynomial by a monomial, as illustrated in the following problem:

**EXAMPLE:**

Divide: $\dfrac{15xyz + 30x^2yz - 25\,xy^2z^2}{5xz}$

Factoring the numerator, the fraction can be rewritten:

$$\frac{5xz(3y + 6xy - 5y^2z)}{5xz}$$

Using the identity property, this equals $3y + 6xy - 5y^2z$.

***Solution of Linear Equations***

Using the skills of working with polynomial expressions and the real number properties outlined in this section, the students should have developed the algebraic operations used to solve any linear equation. The illustration that follows combines most of the properties discussed above.

**EXAMPLE:**

Solve: $3\left(x + \frac{1}{3}\right) = 28$

| | |
|---|---|
| $3(x) + 3\left(\frac{1}{3}\right) = 28$ | Distributive Property |
| $3x + 1 = 28$ | Multiplicative Inverse |
| $(3x + 1) + (-1) = 28 + (-1)$ | Addition Property of Equality |
| $3x + [1 + (-1)] = 27$ | Associative Property |
| $3x + 0 = 27$ | Additive Inverse |
| $3x = 27$ | Property of 0 |
| $\frac{1}{3}(3x) = \frac{1}{3}(27)$ | Multiplication Property of Equality |

| | | | |
|---|---|---|---|
| $1x$ | $=$ | $9$ | Multiplicative Inverse |
| $x$ | $=$ | $9$ | Property of 1 |

## EXTENSION OF ALGEBRAIC SKILLS

After a brief review of algebraic skills covered in Course I, with an emphasis on the properties of the real numbers being used, the students should be ready to strengthen and extend their facility with algebra.

### *Factoring*

The work of this section is a continuation of previous work with factoring. The students should be able to remove a common monomial factor and continue to factor the resulting polynomial, using the techniques of factoring trinomials and differences of two squares.

**EXAMPLE 1:**

$$x^3 + 7x^2 + 12x = x(x^2 + 7x + 12)$$
$$= x(x+3)(x+4)$$

**EXAMPLE 2:**

$$4g^2h^3 - 36g^2h = 4g^2h(h^2 - 9)$$
$$= 4g^2h(h+3)(h-3)$$

### *Simplifying Algebraic Fractions with Polynomial Denominators*

Simplifying fractions should be extended to the case where the numerator and denominator share common factors other than the denominator itself. By using the identity property of 1, the fraction can be simplified.

$$\frac{ab+ac}{ad} = \frac{a(b+c)}{ad} = \frac{a}{a} \cdot \frac{b+c}{d} = 1 \cdot \frac{b+c}{d} = \frac{b+c}{d}$$

**EXAMPLE 1:**

Simplify: $\dfrac{15xyz + 30x^2y - 25xy^2z^2}{10xyz}$, $x, y, z, \neq 0$

**EXAMPLE 2:**

If possible, reduce: $\dfrac{x^2-9}{2x+6}$, $x \neq -3$

**EXAMPLE 3:**

Simplify: $\dfrac{4x^2 - 4x - 120}{2x^2 - 50}$, $x \neq 5, -5$

$$\frac{4(x+5)(x-6)}{2(x+5)(x-5)}$$

This fraction can be reduced to $\dfrac{2(x-6)}{x-5}$.

### *Multiplication and Division of Fractions with Polynomial Denominators*

Simplification of algebraic fractions is useful in the process of multiplying or dividing two or more rational fractions. In this process, numerators and denominators are frequently factored before proceeding.

**EXAMPLE 1:**

Express the following product in simplest form:

$$\frac{x^2-16}{3x-3} \cdot \frac{3x^2+9x-12}{x^2+8x+16}, \quad x \neq 1, -4$$

Factoring each quotient produces

$$\frac{(x+4)(x-4)}{3(x-1)} \cdot \frac{3(x+4)(x-1)}{(x+4)(x+4)}$$

This reduces to $x - 4$.

**EXAMPLE 2:**

Find this quotient:

$$\frac{x^2+5x+6}{x^2+3x} \div \frac{x^2+7x+10}{2x^2+10x}$$

Since division can be defined in terms of multiplication by the reciprocal

$$a \div b = a \cdot \frac{1}{b},$$

this division problem is equivalent to the multiplication of the first fraction by the multiplicative inverse of the second fraction.

Therefore, the given problem is equivalent to

$$\frac{x^2+5x+6}{x^2+3x} \cdot \frac{2x^2+10x}{x^2+7x+10}.$$

***Addition and Subtraction of Fractions***

The students should be able to add rational fractions with monomial or binomial denominators. They should understand that the procedure is mathematically similar to the addition of two fractions from arithmetic. The multiplicative identity 1 is important in finding the least common denominator for the fractions.

**EXAMPLE 1:**

$$\frac{1}{6} + \frac{1}{8} = \frac{1}{6} \cdot \frac{4}{4} + \frac{1}{8} \cdot \frac{3}{3}$$

$$= \frac{4}{24} + \frac{3}{24}$$

$$= \frac{7}{24}$$

**EXAMPLE 2:**

Add: $\frac{2}{x} + \frac{3}{y^2}$

Both fractions can be rewritten with the same denominator by multiplying the first fraction by 1 in the form of $\frac{y^2}{y^2}$ and the second fraction by 1 in the form of $\frac{x}{x}$.

$$\frac{2}{x} + \frac{3}{y^2} = \frac{2}{x} \cdot \frac{y^2}{y^2} + \frac{3}{y^2} \cdot \frac{x}{x}$$

$$= \frac{2y^2}{xy^2} + \frac{3x}{xy^2}$$

$$= \frac{2y^2+3x}{xy^2}$$

**EXAMPLE 3:**

Subtract: $\frac{12}{x} - \frac{5}{x+1}$

**EXAMPLE 4:**

Add: $\frac{2y}{xy} + \frac{3}{y^2}$

## GROUPS AND FIELDS (OPTIONAL)

The concepts of *group* and *field* are very important in the study of mathematical structure. The students often find this topic different, interesting, and enjoyable. The teacher is encouraged to introduce the idea of group with further exploration dependent on time available.

### *Groups*

A group is a set together with a binary operation satisfying the following properties:

i. the set is closed with respect to the operation
ii. the operation is associative on the set
iii. there is an identity element for the operation in the set
iv. every element in the set has an inverse with respect to the operation.

Note that the operation does not have to be commutative. If it is, then the group is *commutative* or *Abelian*. Most of the groups accessible to students at this point are commutative.

An infinite group is one in which the set of elements is infinite. Generally, the infinite groups with which students will work are the familiar ones: reals, rationals, integers, even integers, etc.

Infinite systems that are groups include:

i. reals, rationals, and integers, with addition
ii. positive reals and rationals, with multiplication
iii. integral multiples of any natural number, n, with addition (examples: even numbers, multiples of 7)
iv. Integral powers of any natural number n, with multiplication. (for example, integral powers of 2: the identity is $2^0 = 1$, the inverse of $2^5$ is $2^{-5}$, etc.)

Infinite systems that fail to be groups include:

i. whole numbers, with addition (inverse property fails)
ii. integers, with multiplication (inverse property fails)
iii. odd integers, with addition (closure property fails; also, no identity or inverses)

Since the set of polynomials (under addition) satisfies the closure and associative properties, and since an additive identity and additive inverse exist, they form a group. Also, since the set of polynomials is commutative with respect to addition, they form a commutative or Abelian group.

A finite group is one in which the set of elements is finite. The set T = {0, 2, 4, 6, 8}, together with the operation $\oplus$ defined such that the sum of any two elements is the units digit of the sum under ordinary addition (See table below), forms a group.

| $\oplus$ | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| 0 | 0 | 2 | 4 | 6 | 8 |
| 2 | 2 | 4 | 6 | 8 | 0 |
| 4 | 4 | 6 | 8 | 0 | 2 |
| 6 | 6 | 8 | 0 | 2 | 4 |
| 8 | 8 | 0 | 2 | 4 | 6 |

Verification:

Closure: All sums are in the set
Associativity: $(2 \oplus 4) \oplus 6 = 6 \oplus 6 = 2$
$2 \oplus (4 \oplus 6) = 2 \oplus 0 = 2$, etc.

Identity Element: 0

Inverse:
$2 \oplus 8 = 0$ 8 is the inverse of 2
$4 \oplus 6 = 0$ 4 is the inverse of 6
$6 \oplus 4 = 0$ 6 is the inverse of 4
$8 \oplus 2 = 0$ 2 is the inverse of 8

Other finite systems that can be used to illustrate groups include:

(1) Modular systems, including clock arithmetic, under addition
(2) Modular systems without zero under multiplication if and only if the number of elements is prime
(3) Permutations of a finite set of elements with the operation "followed by"
(4) Rotations of a regular polygon (for example, the rotations of 0°, 120°, 240° about the center of an equilateral triangle)
(5) Any number of "arbitrary" systems, for example, the system consisting of set S with elements ♡, ~, ⊻ and operation ♂

| ♂ | ♡ | ~ | ⊻ |
|---|---|---|---|
| ♡ | ♡ | ~ | ⊻ |
| ~ | ~ | ⊻ | ♡ |
| ⊻ | ⊻ | ♡ | ~ |

Additional topics for more talented students might include investigation of a noncommutative group (for example the permutation group of six elements—which is equivalent to the group of the six symmetries of an equilateral triangle), or, for example, a demonstration that every group of four or fewer elements must be commutative.

Besides these ideas, more capable students can work on the proofs of simple group theorems. All students should appreciate the fact that a major reason for the study of group structure is that any fact proven for a group in general is true for any particular system under investigation, if that system can be shown to be a group.

An important aspect of any mathematical study is efficient notation. A useful notation for the general group is (G, *), symbolizing the group whose set is G and whose operation is *. Thus, the rationals with addition can be called (Q, +), and the system referred to in Ex. 5, above, (S, ♂ ). Also, the set of clock n numbers is usually symbolized, "$Z_n$."

***Fields***

So far, the systems discussed in this section have included one operation. Most situations the students encounter involve more than one operation; usually addition and multiplication. It is reasonable, then, to study the structure of systems consisting of a set together with two operations.

A field is a set together with two operations such that:

i. the set forms a commutative group with identity called zero under the first operation;
ii. the non-zero elements of the set form a commutative group under the second operation;
iii. the second operation distributes over the first.

Examples of systems that are fields include: (Note that "0" is excluded from the multiplicative group.)

i. the reals and rationals, each under addition and multiplication
ii. the clock n systems under addition and multiplication if and only if n is prime

iii. the set {0, 2, 4, 6, 8} under the operations $\oplus$ and $\odot$ defined so that the result of each operation is the units digit of the result obtained under ordinary addition and multiplication.

Examples of systems that are not fields include:

i. The integers, under addition and multiplication . (The inverse property fails for the multiplicative component.)
ii. The clock n systems for n not prime. (At least one non-zero element would fail to have a multiplicative inverse.)

The field structure has many applications in mathematics. Indeed, most of the study of arithmetic and algebra from elementary school onward is related to the fact that the real numbers, with addition and multiplication, are a field.

One of the most important field theorems for secondary school mathematics is:

$$(ab = 0) \leftrightarrow (a = 0) \vee (b = 0)$$

This theorem is the basis for solution of quadratic equations by factoring, as typified by the following example.

$$x^2 + 7x + 12 = 0$$

$$(x + 3)(x + 4) = 0$$

$$x + 3 = 0 \text{ or } x + 4 = 0$$

$$x = -3 \text{ or } x = -4$$

Even the rewriting of the sum as a product is dependent upon the associative and distributive properties of a field. All factoring, as we have seen, depends on appropriate use of the distributive property.

A useful general notation for the field with set F, a first operation corresponding to addition, and a second to multiplication, is the ordered triple $(F, \oplus, \odot)$.

Additional topics might include the proof of simpler field theorems, and, for more advanced students, an investigation of *ordered* fields.

# III.

# *Euclidean Geometry*

The emphasis of this unit is on the proof of many geometric concepts and relationships developed intuitively in Course I and on those which are extended and introduced in Course II. Geometric principles and ideas will be developed and extended in the areas of congruence, similarity, parallelism and inequalities. When possible, concepts from algebra and transformations should be integrated with geometric principles in the solution of problems. The notion of formal geometric proof is an extension of the development of proof in the Logic unit. No specific approach to the study of Euclidean Geometry is mandated. Some teachers may use a transformation approach; others may use a more traditional approach.

## AXIOMATIC SYSTEMS

### *Undefined Terms and Definitions*

Not every term in a logical system can be defined in terms of previously defined terms. In Euclidean geometry the terms POINT, LINE, and PLANE are commonly accepted without definition and are used as building blocks in defining many other terms. The notion of betweenness may be defined or accepted as an undefined term. The students should understand that good definitions use only terms previously defined or accepted as undefined, contain a minimum of necessary information, and must be reversible. Any definition can be stated as a true biconditional; that is, using the "if and only if" connective.

### *Assumptions*

It should be understood that most geometries are postulational systems and depend upon some assumptions (postulates or axioms) whose truth is accepted without proof. The students will use the axioms of the real number system in conjunction with postulates concerning geometric relationships to demonstrate that statements are true.

### *Theorems and Proof*

A theorem is a statement which can be demonstrated to be true using deductive reasoning, i.e., using reasoning patterns developed in the Logic unit.

Geometric proofs are constructed in a manner similar to those in logic. The teacher should use the ideas of the Laws of Detachment and Syllogism in developing the techniques for writing DIRECT proofs. INDIRECT proof depends upon the Law of Contrapositive, the Law of Disjunctive Inference, and the Law of Double Negation. Proofs may be constructed in a two column format or in paragraph form. The undefined terms, definitions, postulates, axioms, and previously proved theorems are used as reasons to substantiate each step in the logical development of a proof starting with the given information and leading to the conclusion.

The students should be familiar with many of the relationships regarding triangles, parallel lines, perpendicular lines, triangles, quadrilaterals, etc. from their work in Course I. Concentration in Course II is placed on the proof of the relationships and on applying these relationships to problem solving.

The theorems listed in each section provide a basis for the proof of many originals. No attempt is made to list the definitions and postulates needed to develop a logically sound unit on geometry. Texts in geometry vary in their presentation of an axiomatic system, and the basic notions of geometric proof can be taught in a variety of ways. In most developments, the first theorems in the system deal with angles (right, vertical, complementary, supplementary). The students should be familiar with the term *congruence* as applied to line segments, angles, and triangles. In Course II, the emphasis is on proving triangles congruent and on applying the congruence relationships to other proofs.

The teacher is encouraged to combine geometric concepts and techniques of algebra in the solution of problems. Examples are found in other sections of this *Syllabus/Guide*.

## CONGRUENCE

### *Assumptions*

It is common to postulate that two triangles are congruent if one of the following conditions is met:

(1) Two angles and the included side of one triangle are congruent to two angles and the included side of the other (ASA).

(2) Two sides and the included angle of one triangle are congruent to two sides and the included angle of the other (SAS).

(3) Three sides of one triangle are congruent to three sides of the other (SSS).

Using one of these postulates, the other two can be proved.

### *Isosceles Triangle Theorems*

Isosceles and equilateral triangles play a major role in the development of geometry. Two important theorems (one the converse of the other) are:

(1) If two sides of a triangle are congruent, then the angles opposite these sides are congruent.

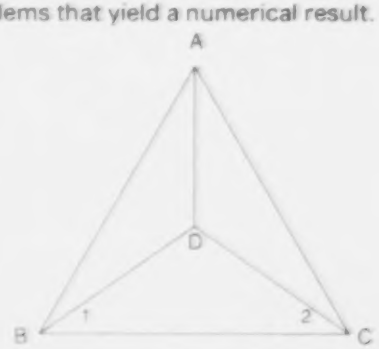
(2) If two angles of a triangle are congruent, then the sides opposite these angles are congruent.

The students will have worked with properties of isosceles and equilateral triangles in Course I. The teacher will find it useful to observe that any isosceles triangle is its own image under a reflection in the line containing the altitude to the base. In Course II, the students should develop original proofs as well as review and extend algebraic techniques used to solve geometric problems that yield a numerical result.

**EXAMPLE:**

Given: $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$
$\overline{AD}$ bisects $\angle BAC$
$\overline{BD}$ and $\overline{CD}$ are drawn

Prove: $\angle 1 \cong \angle 2$



### *Other Congruence Theorems*

Two other methods of proving triangles congruent should be developed:

(1) Two triangles are congruent if two angles and a side opposite one of them are congruent to two angles and the corresponding side in the other (AAS).

(2) Two right triangles are congruent if the hypotenuse and leg of one of them are congruent to the hypotenuse and leg of the other. (HL)

Examples should be given to demonstrate that SSA in general does not establish congruence. It is only in the special case of a right triangle that this combination of parts guarantees congruence and we call this HL.

The postulates and theorems relating to congruent triangles and isosceles triangles should be applied to original exercises and should include proofs with overlapping triangles.

**EXAMPLE:**

Given: $\triangle PQK$ and $\triangle QPM$ with $\angle 1 \cong \angle 2$ and $\overline{MP} \cong \overline{KQ}$

Prove: $\overline{KP} \cong \overline{MQ}$

## INEQUALITIES

As properties of triangles are being developed, attention should be paid to theorems describing inequality relationships. Geometric statements involving inequalities are often proved by means of an indirect proof. This area provides another place to reinforce the concepts of the indirect proof as developed in the Logic unit.

### *Angle Relationships*

Inequality theorems to be included are:

(1) The measure of an exterior angle of a triangle is greater than the measure of either remote interior angle.

(2) If two sides of a triangle are unequal in length, then the angles opposite them are unequal in measure and the larger angle lies opposite the longer side.

### *Relationships of Lengths of Sides*

Inequality theorems to be included are:

(1) If two angles of a triangle are unequal in measure, the sides opposite them are unequal in length and the longer side is opposite the angle with the greater measure.

(2) The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

**EXAMPLE:**

Two sides of a triangle have lengths of 2 and 5. Find all possible lengths of the third side.

## PARALLELISM

### *Definitions and Assumptions*

From their work in Course I, the students should be familiar with terms such as parallel lines, transversals, alternate interior angles, corresponding angles, and interior angles on the same side of the transversal. From Euclid's Parallel Postulate, it is assumed that in a plane one and only one line can be drawn through a given point

parallel to a given line not containing that point. This postulate should be presented and its importance in the development of Euclidean geometry noted.

***Theorems***

The demonstration of the first theorem in the sequence of relationships concerning angles associated with parallel lines is usually an indirect proof. The following theorems should be included:

(1) In a plane, if two lines are cut by a transversal so that the alternate interior angles are congruent, then the lines are parallel.

(2) If two lines are cut by a transversal so that the corresponding angles are congruent, the lines are parallel.

(3) If two lines are cut by a transversal so that the interior angles on the same side of the transversal are supplementary, the lines are parallel.

(4) In a plane, two lines perpendicular to the same line are parallel.

(5) If two parallel lines are cut by a transversal, the alternate interior angles formed are congruent.

(6) If two parallel lines are cut by a transversal, the corresponding angles formed are congruent.

(7) Two lines parallel to a third line are parallel to each other.

(8) In a plane, a line perpendicular to one of two parallel lines is perpendicular to the other.

***Angle Measure in Triangles***

One of the basic theorems of Euclidean geometry states that the sum of the measures of the angles of a triangle is 180°. This theorem is usually demonstrated by drawing a line through one vertex of the triangle parallel to the opposite side of the triangle. This use of Euclid's Parallel Postulate leads to this sum. Other geometries which do not accept Euclid's Parallel Postulate have very different results in this angle measure theorem for triangles.

Many other properties relating to the sum of angle measures of a triangle follow directly. Examples are:

(1) The acute angles of a right triangle are complementary.

(2) The degree measure of each acute angle of an isosceles right triangle is 45.

(3) The degree measure of each angle of an equilateral triangle is 60.

(4) The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

(5) The sum of the measures of the angles of a quadrilateral is 360°.

The general formula for the sum of the measures of the interior angles of a polygon of n sides, $S = (n - 2)180°$, and the sum of the measures of the exterior angles of a polygon, $S = 360°$, follow from this basic theorem. These formulas should be extended to find the measure of each interior angle or each exterior angle of a regular polygon.

***Classification and Properties of Quadrilaterals***

The students should be able to define and classify the following types of quadrilaterals: parallelogram, trapezoid, rectangle, square, and rhombus. The con-

cept of parallelism leads to relatively simple proofs concerning important properties of quadrilaterals. The students should be familiar with the properties below. The teacher should prove some, or all, of these statements.

(1) A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

(2) The opposite sides of a parallelogram are congruent.

(3) The opposite angles of a parallelogram are congruent.

(4) The diagonals of a parallelogram bisect each other.

(5) If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

(6) If two sides of a quadrilateral are both congruent and parallel, the quadrilateral is a parallelogram.

(7) If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram. (Note that this is the converse of (4) above.)

(8) The diagonals of a rhombus are perpendicular.

(9) The diagonals of a rectangle are congruent.

(10) The diagonals of a rhombus bisect the angles of the rhombus.

(11) The base angles of an isosceles trapezoid are congruent.

(12) The diagonals of an isosceles trapezoid are congruent.

## SIMILARITY

Algebraic concepts and properties of ratio and proportion should be reviewed in preparation for the work with similar polygons. The geometric mean, or mean proportional, should be introduced. The Golden Ratio can be discussed here. The teacher will find it useful to relate the notion of dilation to similar triangles.

### *Definition of Similar Polygons*

Polygons are similar if their corresponding angles have equal measure and the lengths of their corresponding sides are proportional.

### *Similar Triangles*

Depending upon the development of the axiomatic system, the following important statement about similar triangles may be postulated or proved: Two triangles are similar if two angles of one are congruent to two angles of the other. Another approach is to postulate or prove: Two triangles are similar if three sides of one are proportional to the corresponding sides of the other.

Other useful theorems to be presented are:

(1) If a line is parallel to one side of a triangle and intersects the other two sides, it divides the other two sides proportionally, and conversely.

(2) If a line segment connects the midpoints of two sides of a triangle, it is parallel to the third side and has a length $\frac{1}{2}$ of the length of the third side.

(3) Two triangles similar to the same triangle are similar to each other.

(4) If an altitude is drawn to the hypotenuse of a right triangle, then

(a) The two triangles formed are similar to the original and to each other.

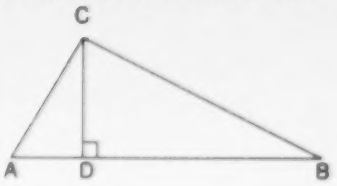
(b) The length of each leg is the geometric mean (mean proportional) between the length of the hypotenuse and the length of the projection of that leg upon the hypotenuse.

(c) The length of the altitude is the geometric mean between the length of the segments of the hypotenuse.

There are many algebraic applications of these properties and theorems. The teacher should include appropriate problems throughout the discussion of similar polygons.

**EXAMPLE 1:**

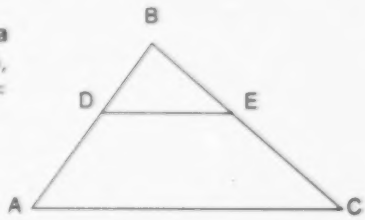
In the diagram, $\triangle ABC$ is a right triangle, $\overline{CD}$ is the altitude to hypotenuse $\overline{AB}$. If $CD = 12$ and DB is seven more than AD, find AD.




**EXAMPLE 2:**

Given: In $\triangle ABC$, D is a point on $\overline{AB}$, E is a point on $\overline{BC}$, $\overline{DE} \parallel \overline{AC}$. $BD = 5$, $DA = x + 2$, $BE = x + 4$, $EC = 2x + 4$.

Find: $x$




### *Pythagorean Relationship*

In Course I, the students worked with the Pythagorean relationship and should be familiar with the theorem asserting that in any right triangle the square of the hypotenuse is equal to the sum of the squares of the legs ( $c^2 = a^2 + b^2$ ). Previous discussions of the relationship have probably been motivated through the use of diagrams involving squares and areas. The theorem can be proved in a great many ways. The most usual approaches are either through the use of the theorem involving the altitude to the hypotenuse of a right triangle or through the use of area relationships.

Many problem-solving activities employ the Pythagorean Theorem. Some of these make use of an algebraic solution and/or properties of the special 30°-60°-90° and 45°-45°-90° triangles.

**EXAMPLE:**

In $\triangle ABC$, $m\angle C = 90$, the length of $\overline{BC}$ is three greater than the length of $\overline{AC}$, and the length of $\overline{AB}$ is two greater than the length of $\overline{BC}$. Find the lengths of the three sides of the triangle.

### *Special Right Triangles*

The use of certain special relationships is advantageous when working with special right triangles, including

(1) knowledge of certain Pythagorean triples: 3-4-5, 5-12-13, 8-15-17, and their multiples

(2) relationships of sides in a 30°-60°-90° triangle

(3) relationships of sides in a 45°-45°-90° triangle

These can be derived from the Pythagorean relationship and/or the properties of isosceles and equilateral triangles.

### *Right Triangle Trigonometry*

When properties of proportions are applied to similar right triangles, the basic notions of right triangle trigonometry can be developed.

The three ratios, sine, cosine, and tangent, should be developed and defined in terms of the hypotenuse and the legs opposite and adjacent to an acute angle of a

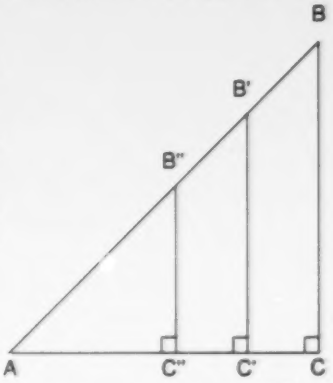
right triangle. The students should understand that these ratios are constant for any acute angle.

In $\triangle ABC$,

$$\text{Sin A} = \frac{\text{length of leg opposite A}}{\text{length of hypotenuse}}$$

$$\text{Cos A} = \frac{\text{length of leg adjacent A}}{\text{length of hypotenuse}}$$

$$\text{Tan A} = \frac{\text{length of leg opposite A}}{\text{length of leg adjacent A}}$$




In the above diagram, since $\triangle ABC \sim \triangle A'B'C' \sim \triangle A'' B'' C''$, $\text{Sin A} = \frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{B''C''}{AB''}$.

Values of the sine, cosine, and tangent of 30°, 45°, and 60° can be derived using the geometric properties of special triangles. The students should learn to use a table of values of the trigonometric ratios. Problems should be considered in which (1) the degree measure of the angle is found, given two sides of the triangle and (2) one side of the triangle is found, given another side and one acute angle of the triangle.

Problems involving the concepts of angle of elevation and angle of depression should be considered.

**EXAMPLE 1:**

Given: Right triangle ABC with right angle C, $AC = 8$, $BC = 5$

Find: m $\angle$ A to the nearest degree.

**EXAMPLE 2:**

The ideal slope for a wheelchair ramp is $\frac{1}{12}$, i.e., for each 12-foot horizontal distance the ramp rises one foot, as shown in the diagram. Find the m $\angle$ A to the nearest degree.

**EXAMPLE 3:**

The angle of elevation of the top of a flagpole from a point on the ground 30 meters from the foot of the pole is 18°. Find the height of the pole to the nearest meter.

## CONSTRUCTIONS AND PROOFS

It is expected that the students will understand the use of the straightedge and compasses from their experiences with them in previous courses. The use of these classic construction instruments should be reviewed, and the students should understand that geometric constructions are done solely through the use of a straightedge to represent straight lines and a pair of compasses to reproduce congruent distances.

The students may be familiar with the procedures for some of the basic constructions presented here. In this course, however, emphasis should be placed on the proof of the constructions.

This unit provides an opportunity for the students to make use of the basic constructions and the intuitive ideas of transformations (symmetry, lines of reflection, etc.) to create artistic designs and/or to use constructions in a problem-solving approach to geometric situations. Optional topics might include a study of the classic impossible constructions, e.g., the trisection of an angle with compass and straight edge and/or projects involving geometric designs.

***Basic Constructions***

The following basic constructions serve as the foundation for more complex geometric constructions.

(1) Construct a line segment congruent to a given line segment.

(2) Construct an angle congruent to a given angle.

(3) Construct the perpendicular bisector of a given line segment.

(4) Bisect a given angle.

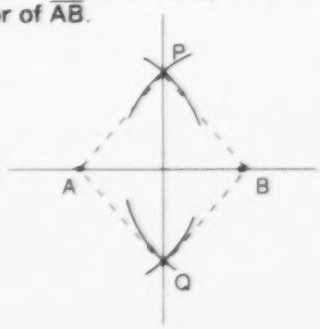
(5) Construct a perpendicular to a given line from a given point not on the line.

(6) Construct a perpendicular to a given line at a given point on the line.

(7) Construct a line parallel to a given line through a given point not on the given line.

As constructions are developed, opportunities arise to formalize some of the intuitive ideas associated with transformations. For example, finding the image of point P reflected in line $\ell$ is equivalent to construction 5 followed by construction 1 listed above.

Discussion should take place demonstrating the validity of each of these basic constructions. For example, in proving the perpendicular bisector construction (#3 above), sets of congruent triangles can be used to demonstrate that the constructed line both is perpendicular to and bisects the given line segment. In the construction below, $\overline{PQ}$ is the perpendicular bisector of $\overline{AB}$.




This construction can also be presented as finding the line of reflection which makes A the image of B or B the image of A.

***Other Constructions***

Using the seven basic constructions, the students should be able to construct geometric figures related to topics studied during various sections of the Geometry units in this course. These include the following:

(1) Construct a triangle, given 3 parts—

a) 3 sides
b) two sides and included angle
c) two angles and included side
d) two angles and side opposite one of them.

(2) Construct a triangle similar to another triangle.

(3) Construct the altitude to any side of a triangle.

(4) Construct the angle bisector of any angle of a triangle.

(5) Construct the median from any vertex of a triangle.

(6) Divide a line segment into a number of congruent segments.

(7) Divide a line segment proportionally.

# IV.

# *Analytic Geometry*

The one-to-one correspondence between the points of the plane and the ordered pairs of real numbers makes it possible to study geometric concepts in terms of numbers and to apply some of the techniques and laws of algebra to the study of geometry. Many of the derivations and proofs used in this section depend on theorems that the students have studied in the section on Euclidean geometry.

## REVIEW OF THE RECTANGULAR COORDINATE SYSTEM

The students should review the terminology and conventions associated with the rectangular coordinate system.

The meaning of the slope of a line should be reviewed. The students should recognize lines with positive or negative slopes, with 0 slope, and with slope undefined. The students should be able to find the slope of a line when the coordinates of any two points on the line are given. They should be able to use slope to determine whether three or more points are collinear.

## PARALLEL AND PERPENDICULAR LINES

### *Parallel Lines*

Two nonvertical lines in the coordinate plane are parallel if and only if their slopes are equal. An informal proof based on similar triangles can be presented for this theorem, which was accepted without proof in Course I.

The students should be able to use slope to prove that two lines or line segments are parallel.

**EXAMPLE:**

Plot the points $(0, -2)$, $(6, 1)$, $(3, 3)$, and $(-1, 1)$. Identify the quadrilateral formed when the four points are joined in order, and justify your answer.

### *Perpendicular Lines*

Two nonvertical lines in the coordinate plane are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other (or the product of their slopes is $-1$). A demonstration of this theorem can be given using two lines that intersect at the origin without loss of generality.

**EXAMPLE 1:**

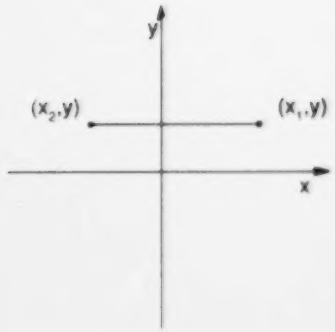
Show that the points $(1, 2)$, $(5, 8)$, and $(-1, 12)$ are the vertices of a right triangle.

**EXAMPLE 2:**

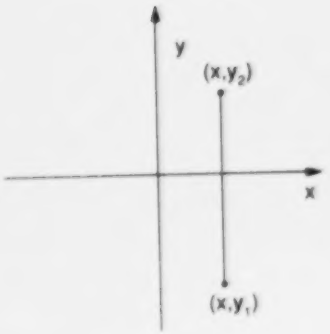
Show that the figure formed by joining the points $(1, -2)$, $(3, 3)$, $(-2, 5)$, and $(-4, 0)$ has perpendicular diagonals.

### *Length of a Segment*

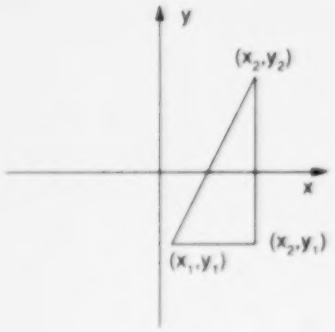
The length of a segment parallel to the x-axis is defined as the absolute value of the difference of the x coordinates (abscissas) of the endpoints of the segment, $|x_2 - x_1|$.




The length of a segment parallel to the y-axis is defined as the absolute value of the difference of the y coordinates (ordinates) of the endpoints of the segment, $|y_2 - y_1|$.




The rectangular coordinate system then makes it possible to use the Pythagorean Theorem to derive a formula for the length of a segment with any two points as endpoints.




$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Line segments in the coordinate system can be shown to be congruent by determining that their lengths are equal.

**EXAMPLE 1:**

Show that the triangle whose vertices are $(-3, 0)$, $(1, -2)$, and $(-1, 4)$ is isosceles.

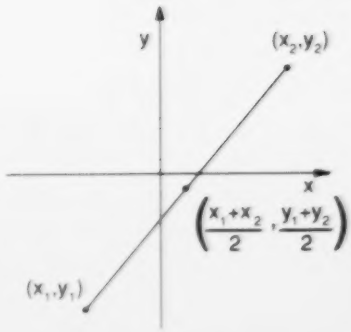
**EXAMPLE 2:**

Plot the points $(-1, 1)$, $(2, 5)$, $(-2, 8)$, and $(-5, 4)$. Show that the figure formed by joining these points in order is a rhombus.

### *Midpoint of a Segment*

The derivation of the formulas for the coordinates of the midpoint of a line segment is based on the assumption from Euclidean geometry that if a line is parallel to one side of a triangle and intersects the other two sides, it divides those sides into segments whose lengths are in proportion.

That $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ are the coordinates of the midpoint of the segment whose endpoints are $(x_1, y_1)$ and $(x_2, y_2)$ can be shown by using the formula for the length of a line segment. The distances from each endpoint to the midpoint can be shown to be equal to each other and equal to one-half the length of the line segment.




**EXAMPLE 1:**

Show that the diagonals of the parallelogram whose vertices are (1, 2), (5, 7), (3, 10), and (−1, 5) bisect each other.

**EXAMPLE 2:**

The coordinates of the midpoint of $\overline{AB}$ are (−2, 5). If the coordinates of A are (1, 1), find the coordinates of B.

## EQUATION OF A LINE

### *Point-slope Form*

The definition of slope makes it possible to write the point-slope form of the equation of a line.

If m represents the slope of a line and (a, b) a point on that line, then by definition, for any other point (x, y) on the line

$$\frac{y - b}{x - a} = m.$$

Therefore, every point on the line can be represented by the equation

$$y - b = m(x - a).$$

### *Slope-intercept Form*

The y-intercept of a line is the y-coordinate of the point at which the line intersects the y-axis. If the y-intercept is b, then (0, b) is a point on the line. Using this special point, the slope-intercept form of the equation of a line can be derived in the same way as was the point-slope form.

Therefore,

$$\frac{y - b}{x - 0} = m$$

or $y - b = mx$

$y = mx + b$

The students should be able to write an equation of a line if the slope and y-intercept are given, or if two points are given, and should be able to determine the slope and y-intercept of a line from its equation.

**EXAMPLE 1:**

A line through (−2, 3) has slope $\frac{1}{2}$. Write an equation of the line.

**EXAMPLE 2:**

Write an equation of the line through (−2, 3) that is

a) parallel to $2x - y = 5$.

b) perpendicular to $2x - y = 5$.

**EXAMPLE 3:**

Write an equation of the line containing (1, 3) and (−2, 6).

## AREA

### *Review of Formulas*

The formulas for the area of a triangle, rectangle, square, parallelogram, and trapezoid should be reviewed.

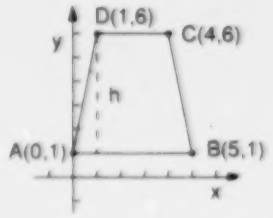
### *Application to Coordinate Geometry*

Finding the area of a polygon for which the coordinates of the vertices are given can provide significant practice with the formulas for length of a line segment, coordinates of the midpoint of a line segment, and slope. Areas can be determined by formula, by dividing the polygon into simpler polygons, or by enclosing the polygon in a larger but simpler figure.

The following examples illustrate the use of these approaches. The students should be encouraged to choose, in each case, the method that is simplest for them.

**EXAMPLE 1:**

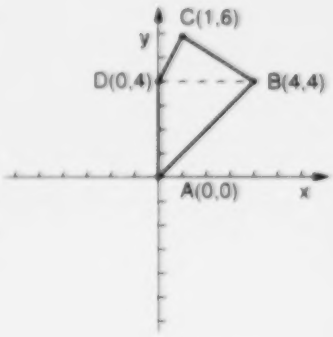
Find the area enclosed when A(0, 1), B(5, 1), C(4, 6), and D(1, 6) are joined in order.




The figure formed is a trapezoid. To use the formula for the area, $A = \frac{1}{2}h(b_1 + b_2)$, it is necessary to find the length of an altitude and of each of the bases. When the bases are not horizontal or vertical lines, this method may be a difficult approach.

**EXAMPLE 2:**

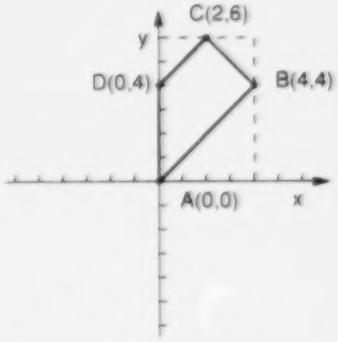
Find the area enclosed when A(0, 0), B(4, 4), C(1, 6), and D(0, 4) are joined in order.




Draw $\overline{DB}$ and find the areas of triangles ABD and DBC. The area of the quadrilateral is the sum of the areas of the triangles into which the quadrilateral is separated. This method may not be possible if the area of the triangles into which the figure is divided cannot be easily determined.

**EXAMPLE 3:**

Find the area enclosed when A(0, 0), B(4, 4), C(2, 6), and D(0, 4) are joined in order.



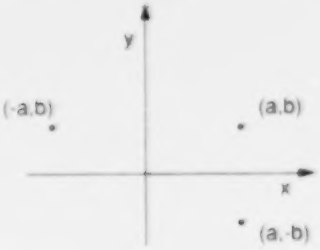

Draw vertical lines through B and D and draw horizontal lines through A and C. From the area of the rectangle that encloses the trapezoid, subtract the areas of the triangles.

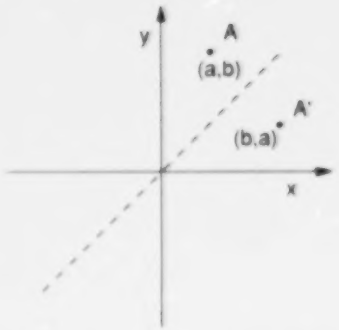
### *Reflection in a Line*

The understanding of a line reflection can be made more concrete for students by applying to the coordinate plane the intuitive concepts learned in Course I. The students should be able to write the coordinates of a point or set of points reflected in the x-axis, the y-axis, or the line whose equation is $y = x$. The use of any other line as a line of reflection is optional.

If A′ is the image of A under a line reflection, then $\overline{AA'}$ is perpendicular to the line of reflection at the midpoint of $\overline{AA'}$; that is, the line of reflection is the perpendicular bisector of $\overline{AA'}$.

Facts that the students have learned about slope and midpoint can be used to demonstrate that $(-a, b)$ is the image of $(a, b)$ under a reflection in the y-axis, that $(a, -b)$ is the image of $(a, b)$ under a reflection in the x-axis, and that $(b, a)$ is the image of $(a, b)$ under a reflection in the line whose equation is $y = x$.




**EXAMPLE:**

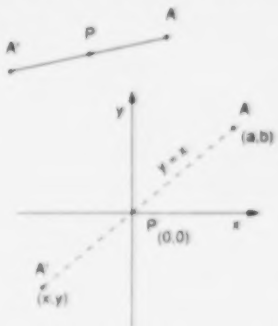
Triangle ABC has coordinates A (0, 1), B (2,5), and C (2,4). State the coordinates of its image under the following reflections:

a) in the x-axis
b) in the y-axis
c) in the line whose equation is $y = x$.

Mathematicians are now investigating the ideas of line reflections to control noise pollution, using mirror image soundwaves or "antinoise" to cancel unwanted noises. NASA is pioneering this technique for use in quieting certain aircrafts which are fuel-efficient but extremely noisy.

### *Reflection in a Point*

If A′ is the image of A under a reflection in the point P, then P is the midpoint of $\overline{AA'}$. It can be shown that $(-a, -b)$ are the coordinates of the image of (a,b) under the reflection in the origin since (0,0) are the coordinates of the midpoint of the segment joining the original point and its image.



Let the coordinates of A′ be (x,y).

Since P is the midpoint of $\overline{AA'}$,

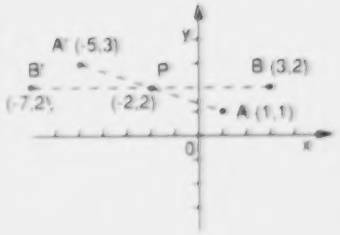
$$\frac{a+x}{2} = 0 \text{ and } \frac{b+y}{2} = 0$$

$\therefore x = -a$ and $y = -b$, or

the coordinates of A′ are $(-a, -b)$.

**EXAMPLE:**

The image of A(1, 1) under a reflection in a point P is A′(−5, 3). Find B′, the image of B(3, 2) under a reflection in P.




Find the coordinates of P, the midpoint of $\overline{AA'}$.

$$\frac{1 + (-5)}{2} = -2 \text{ and } \frac{1 + 3}{2} = 2$$

The coordinates of P are (−2, 2).

Find B′ such that P is the midpoint of $\overline{BB'}$.

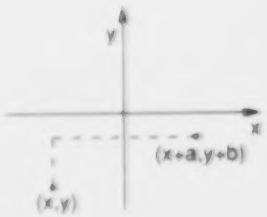
$$\frac{x + 3}{2} = -2 \text{ or } x = -7$$

$$\frac{y + 2}{2} = 2 \text{ or } y = 2$$

The coordinates of B′ are (−7, 2).

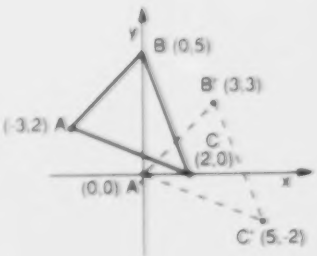
***Translations***

A translation in the coordinate plane is usually expressed in terms of a shift in the horizontal or x direction and/or a shift in the vertical or y direction. If the translation shifts a point a units in the x direction and b units in the y direction, the image (x, y) is (x + a, y + b).




**EXAMPLE:**

The vertices of △ ABC are A(−3, 2), B(0, 5), and C(2, 0). If a translation moves A to (0, 0), what will be the coordinates of the other points?




Since the translation moved (−3, 2) to (0, 0), each point was moved 3 units in the x direction and −2 units in the y direction.

The x coordinate of B′ is 0 + 3 or 3.

The y coordinate of B′ is 5 + (−2) or 3.

The image of B is (3, 3).

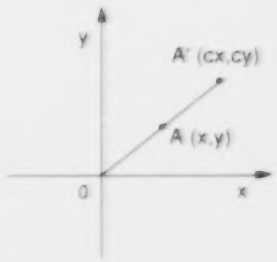
The x coordinate of C′ is 2 + 3 or 5.

The y coordinate of C′ is 0 + (−2) or −2.

The image of C is (5, −2).

### *Dilations*

A dilation with respect to O (the origin) stretches $\overline{OA}$ to $\overline{OA'}$ by a constant c ($c > 0$). If the coordinates of A are (x, y), then the coordinates of A′ are (cx, cy). If $c > 1$, then $OA' > OA$, but if $c < 1$, then $OA' < OA$. If $c = 1$, then $OA' = OA$, and is called the identity dilation.




**EXAMPLE:**

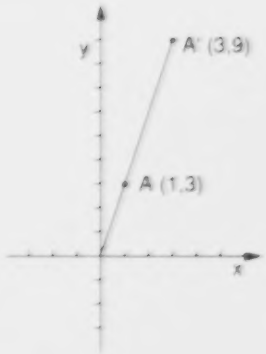
Under a dilation with respect to the origin, the image of A(1, 3) is A′ (3, 9). What are the coordinates of B′, the image of B(2, 1) under the same dilation?

Since $3 = 3(1)$ and $9 = 3(3)$, the constant of the dilation is 3.

The x coordinate of B′ is 3(2) or 6.

The y coordinate of B′ is 3(1) or 3.

The coordinates of B′ are (6, 3).

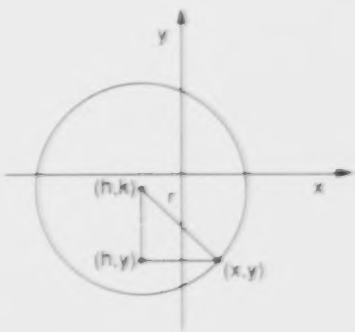



## CONICS

### *Circle*

The derivation of the equation of a circle requires the use of the Pythagorean Theorem or the distance formula. The equation $(x - h)^2 + (y - k)^2 = r^2$ is true for any point (x, y) that is r units from (h, k). The students should be able to write the equation of a circle with a given center and radius, and should be able to determine the radius and center of a circle whose equation is in the form given above.

The teacher should use the ideas of transformations to investigate relationships between two circles. For example, the circle $(x - 2)^2 + (y + 4)^2 = 25$ is a translation of the circle $x^2 + y^2 = 25$, two units to the right and four units down. Likewise, the circle $x^2 + y^2 = 100$ is a dilation of $x^2 + y^2 = 25$, with a constant of dilation 2.




### *Parabola*

The students should recognize an equation in the form $y = ax^2 + bx + c$, $a \neq 0$ as the equation of a parabola, and be able to form a table of values in order to sketch its graph. The parabola is symmetric with respect to the line whose equation is $x = -\frac{b}{2a}$.

Hence, this line is called the axis of symmetry. This value of x is the abscissa of the vertex (turning point) of the parobola. It provides a point of reference for choosing

the x-coordinates of the points to be plotted. It should be noted that in the parabola $y = ax^2 + bx + c$, the constant c is the y intercept.

**EXAMPLE:**

Sketch the graph of $y = x^2 + 4x - 2$. Choose values of x on either side of the axis of symmetry.

$$x = \frac{-b}{2a}$$

$$x = \frac{-(+4)}{2(1)}$$

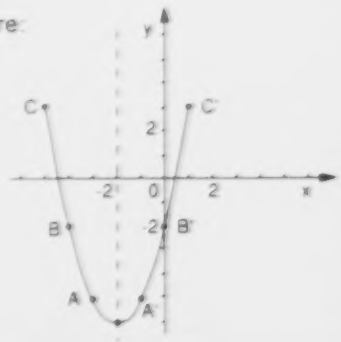
$$x = -2$$

Therefore, a convenient choice of values for x would be from $x = -5$ to $x = 1$.

Two approaches to graphing the parabola are:

a) Make a table of values using all integral values of x from $x = -5$ to $x = 1$.

b) Find the turning point and the axis of symmetry. For each point located on the parabola, find its image point when reflected in the axis of symmetry.




The students should recognize that if $a < 0$, the curve opens downward and has a maximum value, and that if $a > 0$ the curve opens upward and has a minimum value.

The ideas of translation and reflection can be used to advantage in investigating the parabola. For example, $y = x^2 + 2$ is a translation of $y = x^2$; and $y = -x^2$ is a reflection of $y = x^2$ in the x-axis.

### *Other Conics (Optional)*

The teacher may wish to discuss the other conics at this time in order to give the students a more complete presentation of this topic.

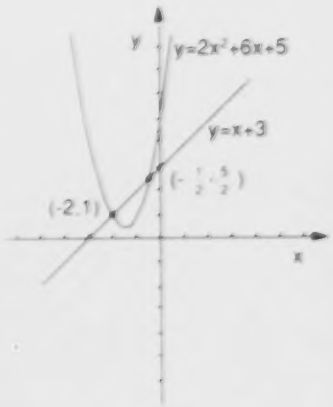
### *Graphic Solution of Systems of Equations*

The simultaneous solution of a quadratic-linear pair of equations can be found by graphic and algebraic methods. For graphic solutions, the second degree equation should be limited to that of a circle or a parabola.

**EXAMPLE:**

Solve $y = 2x^2 + 6x + 5$

$y = x + 3$




**Solution:**

The solution is $\left(-\frac{1}{2}, \frac{5}{2}\right)$, $(-2, 1)$

Compare this with the algebraic solution shown in section V.

The graphic solution of $ax^2 + bx + c = 0$ is a special case of the solution of a quadratic-linear pair for which the quadratic equation is $y = ax^2 + bx + c$ and the linear equation is $y = 0$ (the x-axis).

## LOCUS

### *Definition*

A locus is the set of all points and only those points that satisfy a given condition or conditions. (A point belongs to a locus if and only if it satisfies the given condition or conditions.) In this course, only loci in the plane are required. Teachers may wish to extend these concepts to three dimensions.

### *Basic Loci*

The students should recognize the following basic loci. The locus of points:

a. at a fixed distance from a fixed point
b. at a fixed distance from a fixed line
c. equidistant from two fixed points
d. equidistant from two fixed parallel lines
e. equidistant from two fixed intersecting lines

The students should be able to describe the locus in words and, when the coordinates of the fixed points or the equation of the fixed lines are given, be able to write the equation or equations of the locus.

**EXAMPLE 1:**

Write an equation of the locus of points equidistant from $y = 2x - 2$ and $y = 2x + 6$.

**EXAMPLE 2:**

Write an equation of the locus of points 4 units from $(-1, 3)$.

**EXAMPLE 3:**

Write an equation of the locus of points equidistant from $(1, 3)$ and $(1, 7)$.

**EXAMPLE 4:**

Write an equation of the locus of points equidistant from $(-1, 3)$ and $(7, 7)$.

### *Compound Locus*

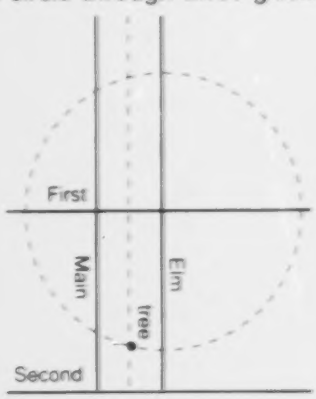
The students should be able to determine a locus that is the combination of these basic loci.

**EXAMPLE 1:**

The locus of points equidistant from three non-collinear points is a point. This idea can be used to find the center of a circle through three given points.

**EXAMPLE 2:**

The town park is bounded by Elm and Main Streets which are parallel to each other and 100 meters apart, and by First and Second Streets, which are perpendicular to Elm and are 300 meters apart. The maintenance crew is instructed to plant a tree equidistant from Elm and Main and 200 meters from the corner of Elm and First. Locate the place where the tree is to be planted.



### *A Special Locus*

The locus of points equidistant from a fixed point and a fixed line is a parabola. The derivation of an equation such as $4y = x^2$ for which the fixed point (focus) is (0, 1) and the fixed line (directrix) is $y = -1$ is a relatively simple classroom derivation. Such derivations should be limited to those in which the fixed line is parallel to the x-axis or y-axis.

## ANALYTIC PROOF

Coordinate geometry furnishes new techniques for geometric proof. The students' introduction to analytic proof should follow from experience with numerical proofs. Many examples similar to the following have been given in this section.

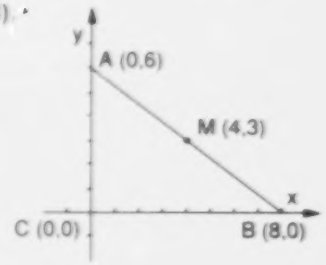
**EXAMPLE 1:**

Show that the midpoint of the hypotenuse of the right triangle whose vertices are A(0, 6), B(8, 0), and C(0, 0) is equidistant from the vertices.

**Solution:**

The midpoint of the hypotenuse $\overline{AB}$ is M(4, 3).

$$CM = \sqrt{(0-4)^2 + (0-3)^2}$$
$$= \sqrt{16+9} = 5$$
$$AM = \sqrt{(0-4)^2 + (6-3)^2}$$
$$= \sqrt{16+9} = 5$$
$$BM = \sqrt{(8-4)^2 + (0-3)^2}$$
$$= \sqrt{16+9} = 5$$




Therefore M is equidistant from the vertices.

It is often possible when setting up a general proof to choose the position of the figure in the coordinate system in a way that simplifies computation. Such a choice should not detract from the universality of the proof.

**EXAMPLE 2:**

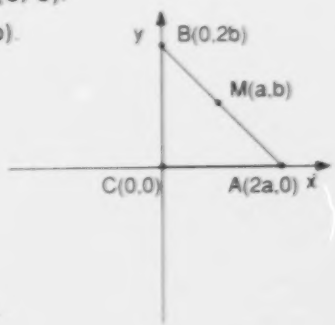
Prove that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices.

Proof: Since the right triangle can be translated, rotated, and/or reflected; it can be positioned with the vertex of the right angle at the origin so that the legs of the triangle are segments of the x and y axis.

Let the vertices be A(2a, 0), B(0, 2b), and C(0, 0).

The midpoint of the hypotenuse $\overline{AB}$ is M(a, b).

$$CM = \sqrt{(0-a)^2 + (0-b)^2}$$
$$= \sqrt{a^2 + b^2}$$
$$AM = \sqrt{(2a-a)^2 + (0-b)^2}$$
$$= \sqrt{a^2 + b^2}$$
$$BM = \sqrt{(0-a)^2 + (2b-0)^2}$$
$$= \sqrt{a^2 + b^2}$$




Therefore M is equidistant from the vertices.

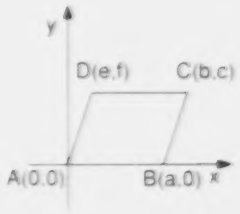
Notice that the use of coordinates 2a and 2b for the vertices makes it possible to avoid the use of fractions for the coordinates of the midpoint.

**EXAMPLE 3:**

Prove that if two sides of a quadrilateral are parallel and congruent, then the quadrilateral is a parallelogram.

Proof: Choose one vertex at the origin and choose as one side a segment of the x-axis.

Given: $\overline{AB} \parallel \overline{DC}$
$\overline{AB} \cong \overline{DC}$




Slope of $\overline{AB} = 0$

Slope of $\overline{DC} = \frac{c - f}{b - e} = 0$

Therefore $c - f = 0$
$f = c$

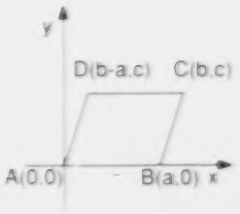
$AB = a - 0 = a$

$DC = b - e = a$

Therefore, $e = b - a$, and D can be rewritten as $(b - a, c)$.

Slope of $\overline{BC} = \frac{c - 0}{b - a} = \frac{c}{b - a}$

Slope of $\overline{AD} = \frac{c - 0}{b - a - 0} = \frac{c}{b - a}$




Lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are parallel because their slopes are equal. ABCD is a parallelogram because the quadrilateral has two pairs of parallel sides.

# V.

# *Equations and Inequalities*

The placement of the solution of equations in a unit that follows those on geometry and analytic geometry is not intended to suggest that all work with equations follows the work with geometry nor that all work with equations be done at one time during the course. The material in this section should be presented to students throughout the course when the solution of an equation will effectively illustrate a concept or solve a problem that arises in the study of the material of the other units.

**REVIEW**

## *Linear Equations and Inequalities*

The techniques for solving linear equations and inequalities which were presented in Course I should be reviewed and used throughout Course II. The level of difficulty of the equations and inequalities should gradually be increased. The problems encountered in the study of Euclidean and analytic geometry and of probability and combinatorics will provide many applications.

**EXAMPLE 1:**

In parallelogram ABCD, m $\angle$ A is 10 degrees more than m $\angle$ B. Find the measure of each angle of the parallelogram.

**Solution:**

Since the consecutive angles of a parallelogram are supplementary, the sum of the measures of angles A and B is 180 degrees.

$$\begin{aligned} \text{Let} \quad x &= m\angle B \\ x + 10 &= m\angle A \\ x + x + 10 &= 180 \\ 2x + 10 &= 180 \\ 2x &= 170 \\ x &= 85 \end{aligned}$$

Therefore, the measure of $\angle$ A is 95°
the measure of $\angle$ B is 85°.

**EXAMPLE 2:**

The line whose equation is $2x + ky = k - 2$ contains the point $(2, -1)$. Find the value of k.

**Solution:**

$$\begin{aligned} 2x + ky &= k - 2 \\ 2(2) + k(-1) &= k - 2 \\ 4 - k &= k - 2 \\ -2k &= -6 \\ k &= 3 \end{aligned}$$

**EXAMPLE 3:**

The measure of one acute angle of an obtuse triangle is 10 degrees more than the measure of the other acute angle. What are the possible degree measures of the smaller acute angle?

**Solution:**

Since the triangle is obtuse, the sum of the measures of the acute angles is less than 90 degrees.

$$\begin{aligned} \text{Let } x &= \text{the measure of the smaller acute angle} \\ x + 10 &= \text{the measure of the larger acute angle} \\ 2x + 10 &< 90 \\ x &< 40 \end{aligned}$$

The measure of the smaller acute angle is less than 40 degrees.

***Proportions***

The meaning, terminology, and solution of proportions should be reviewed before the geometry of similar triangles is presented. The students should be able to identify the extremes and means of a proportion, and be able to find the mean proportional (geometric mean) between two numbers. The problems encountered in the study of Euclidean and analytic geometry and of probability and combinatorics should provide many applications of these ideas.

**EXAMPLE 1:**

A sack contains only red disks and green disks. The number of green disks is five more than the number of red disks. If the probability of drawing a green disk is $\frac{2}{3}$, how many disks are in the sack?

**Solution:**

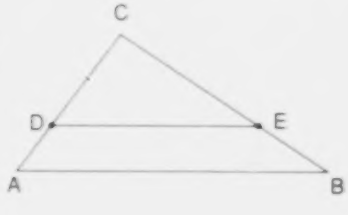
Let

$$\begin{aligned} x &= \text{the number of red disks} \\ x + 5 &= \text{the number of green disks} \\ 2x + 5 &= \text{the total number of disks} \\ \frac{x+5}{2x+5} &= \frac{2}{3} \\ 3(x+5) &= 2(2x+5) \\ 3x + 15 &= 4x + 10 \\ 5 &= x \end{aligned}$$

Therefore in the sack there are 15 disks of which 5 are red and 10 are green.

**EXAMPLE 2:**

In the diagram, $\dot{D}\dot{E} \parallel \dot{A}\dot{B}$, CD = 8, DA = 6, and CE is 6 more than EB. Find CE and EB.




**EXAMPLE 3:**

The altitude to the hypotenuse of a right triangle divides the hypotenuse into segments whose measures are 4 cm and 9 cm. Find the area of the triangle.

## FRACTIONAL EQUATIONS

A fractional equation is one in which the variable occurs in the denominator. The solution of fractional equations other than proportions should be limited to those that lead to equivalent equations that are linear.

**EXAMPLE 1:**

Solve for $x$: $\frac{1}{3} + \frac{1}{x} = \frac{2}{9}$

**EXAMPLE 2:**

If 2 is added to the reciprocal of a number, the result is $\frac{5}{2}$ decreased by the reciprocal of the given number. Find the number.

## QUADRATIC EQUATIONS

The equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is a quadratic equation over the set S when the values of a, b, and c are elements of the set S. The roots of a quadratic equation over the set of rational numbers may be rational, irrational, or imaginary. In Course II, the quadratic equations studied are those over the set of rational numbers that have rational or irrational roots.

### *Solution by Factoring*

If the roots of a quadratic equation are rational, the equation, or an equivalent equation, can be solved by factoring over the set of integers. This method of solution should be reviewed. The basic principle that underlies the solution of a quadratic equation by factoring should be familiar to the students.

$(ab = 0) \leftrightarrow (a = 0) \vee (b = 0)$

The product of two real numbers is 0 if and only if at least one of the factors is 0.

### *Solution by Completing the Square*

If the polynomial $ax^2 + bx + c$ or some multiple of it is not factorable over the set of integers, then the roots of $ax^2 + bx + c = 0$ are not rational, and other techniques are needed to find them.

One such technique is the method of completing the square. This method should be introduced first with simple equations with integral roots and then demonstrated for more difficult ones with irrational roots. The method of completing the square is used to derive the quadratic formula and will be used in future mathematics courses when working with the equations of conics.

**EXAMPLE 1:**

Solve by completing the square.

$x^2 - 6x - 7 = 0$

**Solution:**

$$\begin{aligned} x^2 - 6x &= 7 \\ x^2 - 6x + 9 &= 7 + 9 \\ (x - 3)^2 &= 16 \\ x - 3 &= \pm 4 \\ x - 3 = 4 \quad &\text{or} \quad x - 3 = -4 \\ x = 7 \quad &\text{or} \quad x = -1 \end{aligned}$$

The solution set is $\{7, -1\}$.

Note that this equation could have been solved by factoring.

Equations with irrational roots that cannot be solved by factoring should be demonstrated.

**EXAMPLE 2:**

Solve by completing the square.

$2x^2 - 5x - 1 = 0$

**Solution:**

$$2x^2 - 5x = 1$$

$$x^2 - \frac{5x}{2} = \frac{1}{2}$$

$$x^2 - \frac{5x}{2} + \frac{25}{16} = \frac{1}{2} + \frac{25}{16}$$

$$\left(x - \frac{5}{4}\right)^2 = \frac{33}{16}$$

$$x - \frac{5}{4} = \frac{\pm\sqrt{33}}{4}$$

$$x = \frac{5 \pm \sqrt{33}}{4}$$

### *Solution by Formula*

The repetition of the same steps in the solution of each equation by completing the square should suggest to the students that these steps applied to the general equation $ax^2 + bx + c = 0$ will result in a generalized solution or formula. A derivation of the quadratic formula should be given in class.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = \frac{-c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The students should understand the relationship between the roots of $ax^2 + bx + c = 0$ and the x-coordinates of the points at which the graph of $y = ax^2 + bx + c$ intersects the x-axis.

### *Applications*

The ability to solve quadratic equations by factoring or by the quadratic formula enables the students to develop an algebraic solution for a wide variety of problems encountered in the other sections of this course.

**EXAMPLE 1:**

From the list of members of a chess club, two names are chosen at random to represent the club at a tournament. If there are 78 possible combinations of names that could be chosen, how many names are on the club roster?

**Solution:**

Let $x$ = the number of names on the roster

$${}_xC_2 = 78$$

$$\frac{x(x-1)}{2(1)} = 78$$

$$\frac{x^2 - x}{2} = 78$$

$$x^2 - x = 156$$

$$x^2 - x - 156 = 0$$

$$(x + 12)(x - 13) = 0$$

$$x + 12 = 0 \quad \text{or} \quad x - 13 = 0$$

$$x = -12 \quad \text{or} \quad x = 13$$

We reject the negative root in this case. Therefore, there are 13 names on the club roster.

Note that this quadratic equation could also have been solved by using the quadratic formula.

**EXAMPLE 2:**

The altitude to the hypotenuse of a right triangle separates the hypotenuse into two parts, one of which is 4 cm longer than the other. If the length of the altitude is 6 cm, find the length of the hypotenuse.

**Solution:**

Let $x$ = the length of the shorter part
$x + 4$ = the length of the longer part

$$\frac{x}{6} = \frac{6}{x + 4}$$

$$x(x + 4) = 6^2$$

$$x^2 + 4x = 36$$

$$x^2 + 4x - 36 = 0$$

$$x = \frac{-4 \pm \sqrt{16 + 144}}{2}$$

$$x = \frac{-4 \pm \sqrt{160}}{2}$$

$$x = \frac{-4 \pm 4\sqrt{10}}{2}$$

$$x = -2 \pm 2\sqrt{10}$$

The length of a line segment must be represented by a positive number.

Therefore, $-2 + 2\sqrt{10}$ is the length of the shorter part,
and $-2 + 2\sqrt{10} + 4$ or $2 + 2\sqrt{10}$ is the length of the longer part.

Note that the quadratic equation could not have been solved by factoring.

**EXAMPLE 3:**

The radius of a circle with its center at the origin is 6. Find the coordinates of two points on the circle that have a y-coordinate that is 2 more than the x-coordinate.

**EXAMPLE 4:**

The length of one diagonal of a rhombus is 4 more than the length of the other. The length of one side of the rhombus is 10. Find the lengths of the diagonals.

**EXAMPLE 5:**

Find the positive root, in radical form, of the quadratic equation $x^2 - x - 1 = 0$.

Note that the solution of this equation is the ratio of the sides of the golden rectangle.

## PAIRS OF EQUATIONS

### *Review of Pairs of Linear Equations*

The addition-subtraction method and the substitution method of solving a pair of simultaneous linear equations should be reviewed.

**EXAMPLE 1:**

Two hamburgers and an order of fries costs \$3.60. At the same prices, a hamburger and two orders of fries costs \$3.00. Find the cost of a hamburger and of an order of fries.

**EXAMPLE 2:**

A school purchased three special editions of a textbook. The cost of one textbook in braille and two with large print, for visually impaired students, was \$36.00. At the same prices, one large print textbook and two braille textbooks would cost \$30.00. Find the cost of each type of textbook.

**EXAMPLE 3:**

A sack contains red and green marbles. If one marble is drawn, the probability that it is red is $\frac{3}{4}$. Five red marbles are removed from the sack. Now, if one marble is drawn, the probability that it is red is $\frac{2}{3}$. How many red and how many green marbles were in the sack at the start?

**Solution:**

Let $x$ = the original number of red marbles
$y$ = the number of green marbles

$$\frac{x}{x+y} = \frac{3}{4} \qquad 3x + 3y = 4x$$

$$\frac{x-5}{x+y-5} = \frac{2}{3} \qquad 3x - 15 = 2x + 2y - 10$$

$$-x + 3y = 0$$
$$x - 2y = 5$$
$$y = 5$$
$$x = 15$$

The solution of these two equations tells us that the sack originally contained 15 red and 5 green marbles.

**EXAMPLE 4:**

Find the coordinates of the point on the line $y = 3x - 7$ whose abscissa is 3 more than the ordinate.

### *Quadratic-Linear Pair*

The students should be able to use substitution to find the common solution of two equations in two variables when one is a linear equation and the other is an equation of second degree.

The students should compare the algebraic and graphic solutions of these quadratic-linear systems in which the second degree equation is that of a familiar curve (parabola or circle).

**EXAMPLE:**

Solve the system:

$y = 2x^2 + 6x + 5$
$y - x = 3$

**Solution:**

$y - x = 3$
$y = x + 3$
$x + 3 = 2x^2 + 6x + 5$
$0 = 2x^2 + 5x + 2$
$0 = (2x + 1)(x + 2)$

$2x + 1 = 0$ or $x + 2 = 0$

$2x = -1$ $\qquad$ $x = -2$

$x = -\frac{1}{2}$

$y = x + 3$ $\qquad$ $y = x + 3$

$y = -\frac{1}{2} + 3$ $\qquad$ $y = -2 + 3$

$y = \frac{5}{2}$ $\qquad$ $y = 1$

The solution set is $\left\{\left(-\frac{1}{2}, \frac{5}{2}\right), (-2, 1)\right\}$.

Compare this with the geometric solution shown in Unit IV.

# VI.

# *Probability and Combinatorics*

This unit is a continuation of the study of probability begun in Course I. Its purposes are to review and reinforce understanding of preliminary concepts in probability and combinatorics, to continue work with permutations, and to introduce the idea of *combinations*.

## REVIEW OF FUNDAMENTALS

### *Probability Concepts*

The most basic of all probability concepts is that the probability of an event is equal to the number of ways the event can occur divided by the total number of outcomes, or

$$P(E) = \frac{\text{number of successes}}{\text{total number of outcomes}}$$

This idea can be symbolized:

$$P(E) = \frac{n(E)}{n(S)}$$

where S represents an outcome set and E an event under discussion.

From this fundamental idea, a number of other important principles follow.

1. The probability of an impossible event is 0 (zero), and the probability of a certain event is 1. Thus if E is any event, P(E) can never be negative or exceed 1. That is, for any event E, $0 \leq P(E) \leq 1$.
2. The probability of the complement of an event can be found by subtracting the probability of the event from 1. That is, $P(\sim E) = 1 - P(E)$.
3. P(A or B) = P(A) + P(B) − P(A and B). Using set notation, this becomes:

   $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

   If A and B have no elements in common, then $A \cap B = \emptyset$, and the formulas become

   P(A or B) = P(A) + P(B) and
   $P(A \cup B) = P(A) + P(B)$.

   The Counting Principle should also be reviewed at this time. It states that if one event can occur in any of m ways and another in any of n ways, then the total number of ways both events can occur is given by mn.

A useful extension of the Counting Principle for probability situations (assuming independent events) is:

> If the probability of one event occurring is r and the probability of another occurring is s, then the probability of their both occurring is rs.

Sample spaces and tree diagrams, useful aids for problem solving in probability settings, might also be reviewed.

### *Permutation Concepts*

In Course I, the students learned how to solve basic counting problems involving permutations. The idea of a *permutation*—an arrangement where *order* counts—should be reviewed. The students should have practice finding the number of permutations of n objects taken either all at a time, or taken less than n at a time. The rules given below should be reviewed, and the students should be able to solve problems such as the type indicated.

The number of permutations of n objects taken all at a time, symbolized

$_nP_n$, is given by the formula $_nP_n = n!$.

**EXAMPLE:**

In how many ways can five students be seated in five seats in a row if:

i) there are no restrictions
ii) one of the five, Mary, must sit in the middle.

The number of permutations of n objects taken r at a time ($r \leq n$), symbolized $_nP_r$, is given by the formula

$$_nP_r = n(n-1)(n-2)\cdot\ldots\cdot(n-r+1) = \frac{n!}{(n-r)!}$$

**EXAMPLE:**

Using the digits 1, 2, 5, 6, 8, how many three-digit numbers can be formed if:

i) there are no restrictions
ii) no digit may be repeated
iii) no digit may be repeated and the number must be greater than 500?

## EXTENSION OF WORK WITH PERMUTATIONS

The material just reviewed provides the groundwork for extending the students' understanding of permutations to include cases where the objects to be arranged are not all different.

### *Number of permutations of n objects, r identical, taken all at a time*

It is sometimes difficult for the students to understand the rule governing permutations where some of the elements are identical. A problem-solving approach, similar to that given below, may be a helpful classroom activity. If an approach like this is used, the students should be encouraged to do most of the work.

**PROBLEM:**

Find the number of permutations of the letters in EXETER.

**Solution:**

Suppose that the E's in "EXETER" were differentiable from one another in some manner, say by number. Then, suppose that we try to solve our problem by listing all the permutations of the six "different" letters.

We would have, to start:

$E_1X\,E_2T\,E_3R$ $\quad$ $E_1X\,E_3T\,E_2R$ $\quad$ $E_3X\,E_1T\,E_2R$

$E_2X\,E_1T\,E_3R$ $\quad$ $E_2X\,E_3T\,E_1R$ $\quad$ $E_3X\,E_2T\,E_1R$

and

$E_1E_2E_3X\ T\ R$ $E_1E_3E_2X\ T\ R$ $E_3E_1E_2X\ T\ R$

$E_2E_1E_3X\ T\ R$ $E_2E_3E_1X\ T\ R$ $E_3E_2E_1X\ T\ R$

and so forth.

But the E's in the original problem are not numbered; hence, the first six permutations using subscripted E's are actually the same permutation, after the subscripts are removed. So are the next six. Indeed, every permutation in the final answer to our question would be one of six different "permutations" with the subscripts. This means that the final answer is one-sixth of the total number of permutations with subscripts. The divisor is 6, because, by previous work, the number of ways of permuting the three E's in whatever places they hold is 3!.

This problem, and ones like it, will help to convince the students that the number of permutations of n objects, r of which are identical, taken all at a time, is given by $\frac{n!}{r!}$.

**EXAMPLE 1:**

Find the number of permutations of the letters QBBVRXT.

**Solution:**

There are seven symbols, of which two are identical. Hence the answer is $\frac{7!}{2!} = 2520$.

**EXAMPLE 2:**

What is the number of permutations of these symbols: $$$$$?

**Solution:**

There are five symbols, and all five are identical. Hence, the answer is $\frac{5!}{5!} = 1$.

***Number of permutations of n objects, r identical, s identical, t identical.. taken all at a time.***

This is a logical extension of the previous discussion, and a similar, though more complex, analysis will yield the correct rule for this situation. The number of permutations is given by

$$\frac{n!}{r!\ s!\ t!\ \ldots}$$

**EXAMPLE:**

How many arrangements are there of the letters in MISSISSIPPI?

**Solution:**

There are eleven letters, of which four are of one kind, four of a second kind, two of a third kind. The answer is

$$\frac{11!}{4!\ 4!\ 2!} = 34{,}650.$$

## COMBINATIONS

***Distinction between combination and permutation***

The most important new concept of this unit is the introduction of the idea of *combination*. The distinction between *permutations*, which are *arrangements* where *order counts*, and *combinations*, which are sets, or selections, where order does not count, is frequently difficult for some students to grasp.

Sometimes it is useful to stress that a combination of objects (symbols, people, whatever) corresponds to a *set* of objects. The students know that the elements of the set may be listed in any order, i.e., {A, B, C, D} is the same set as {D, A, C, B}. Sometimes it is helpful to use *committees* to illustrate the meaning of combinations; i.e., Adams, Baker, Cohen is the same committee as Cohen, Adams, Baker.

Permutations, however can be compared to "words". "ACT" is a different word from "CAT." Permutations may also be compared to office-holders in committees. {President Adams, Treasurer Baker, Secretary Cohen} is a different set of officers from {President Cohen, Treasurer Adams, Secretary Baker}.

***Combinations of n objects, taken r at a time, $r \leq n$.***

As previously defined, a combination is a selection where order does not matter; only the identity of the elements is significant. One way to find the number of combinations of three letters which can be chosen from XBLT is to list the three-letter combinations by inspection. The results can be listed:

1. XBL
2. XBT
3. XLT
4. BLT

Thus, there are four such combinations.

There are three notations which are commonly used to indicate the number of combinations of n things taken r at a time. These are

$${}_nC_r \qquad \binom{n}{r} \qquad C(n, r)$$

Generally, in this *Syllabus/Guide*, ${}_nC_r$ notation will be used.

**EXAMPLE:**

Find: i) ${}_4C_3$ ii) ${}_4C_2$ iii) ${}_4C_4$

**Solution:**

i) We have done this above.
$${}_4C_3 = 4$$

ii) This problem can also be solved by listing. If the four objects are A, B, C, D, then the required two-object combinations are:

| | | |
|---|---|---|
| AB | BC | CD |
| AC | BD | AD |

$${}_4C_2 = 6$$

iii) A brief investigation shows that there is only *one* combination of four objects taken four at a time: ABCD. Thus, ${}_4C_4 = 1$.

The problem now arises of determining the value of ${}_nC_r$ for less manageable n's and r's. To develop a general formula for ${}_nC_r$, a comparison of ${}_4P_3$ and ${}_4C_3$ is useful.

It has already been shown that ${}_4P_3$, the number of permutations of four objects—say, XBLT—taken three at a time, is 24.

These 24 permutations are listed below.

| | | | |
|---|---|---|---|
| XBL | XBT | XLT | BLT |
| XLB | XTB | XTL | BTL |
| BLX | TBX | TLX | TLB |
| BXL | TXB | TXL | TBL |
| LBX | BXT | LTX | LBT |
| LXB | BTX | LXT | LTB |

Note that the permutations have been grouped by *combinations*; that is, every listing in each column represents the *same combination*. To find ${}_4C_3$, the total number

of *permutations* of the three objects must be divided by the number of ways which the three objects can be permuted among themselves, 3! or 6. Thus, ${}_4C_3 = \frac{{}_4P_3}{3!} = \frac{24}{6} = 4$.

The generalization of this approach yields

$${}_nC_r = \frac{{}_nP_r}{r!}$$

Equivalent statements of this result are:

$${}_nC_r = \frac{n(n-1)(n-2)\cdot\ldots\cdot(n-r+1)}{r!}$$

$${}_nC_r = \frac{n!}{r!\,(n-r)!}$$

If $r = n$, using the formula ${}_nC_r = \frac{{}_nP_r}{r!}$, ${}_nC_n = \frac{{}_nP_n}{n!} = \frac{n!}{n!} = 1$; and, using the formula ${}_nC_r = \frac{n!}{r!(n-r)!}$, ${}_nC_n = \frac{n!}{n!0!}$. Hence, $\frac{n!}{n!0!} = 1$ and 0! must equal 1. It is evident that every choice of a combination of r things from a total of n things leaves a combination of $n - r$ things unchosen. Hence, it can be seen that ${}_nC_r = {}_nC_{n-r}$.

**EXAMPLE 1:**

Compute ${}_7C_3$.

**Solution:**

$${}_7C_3 = \frac{{}_7P_3}{3!} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!}$$

This equals $\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 4\cdot 3\cdot 2\cdot 1} = 35$

**EXAMPLE 2:**

In how many ways can a committee of 14 senators be chosen from the 100 United States senators?

**Solution:**

$${}_{100}C_{14} = \frac{100!}{14!\,86!}$$

Note that the number named in the answer to Example 2 is large and tedious to compute. The teacher may wish to have the students compute such expressions, but it is suggested that this aspect of the work be deemphasized. The teacher may wish to have the students use calculators to aid them in some of the work of this section.

**EXAMPLE 3:**

Decide if each of the following should be treated as a permutation or a combination problem.

i) How many different bridge hands of 13 cards can be dealt from a standard deck of 52 cards?

ii) A raffle drawing will have three prizes. The first prize is a TV set, the second prize is a camera, and the third prize is a trip to Albany. If 750 people each hold one ticket, how many winning trios are possible?

iii) Same as (ii), but the three prizes are the same.

***Applications***

**EXAMPLE 1:**

If two cards are picked at random from a deck of 52, what is the probability they are both aces?

**Solution:**

Since probability is a ratio, the answer can be obtained using either combinations or permutations. The probability can be expressed as

$$\frac{{}_4C_2}{{}_{52}C_2} \text{ or } \frac{{}_4P_2}{{}_{52}P_2}$$

**EXAMPLE 2:**

A committee has six women and four men. A subcommittee of four is to be chosen by lot.

i) What is the total number of ways the choice can be made?

ii) What is the probability the subcommittee will turn out to consist of four women?

iii) How many different committees of two women and two men are possible?

iv) What is the *probability* the subcommittee will turn out to have two men and two women?

**Solution:**

i) There are 10 people from whom a selection of 4 must be made. Answer: ${}_{10}C_4$

ii) The women can be chosen in ${}_6C_4$ ways. Answer:

$$\frac{{}_6C_4}{{}_{10}C_4}$$

iii) The women can be chosen in any of ${}_6C_2$ ways. For each of these ways, there is one of ${}_4C_2$ selections of two men. We use the Counting Principle to reach the answer: ${}_6C_2 \cdot {}_4C_2$

iv) Using parts (iii) and (i), we have

$$P(2W, 2M) = \frac{{}_6C_2 \cdot {}_4C_2}{{}_{10}C_4}$$

This unit provides abundant opportunities for the students to develop problem-solving skills through the analysis and investigation of some of the special situations which are related to permutations, combinations, and probability.

# Evaluation

The teacher is encouraged to develop his/her own formative (ongoing) evaluation strategies to monitor the students' achievement throughout the year. These strategies should include written and oral testing, small group work assignments, and whole class projects. Examples presented in the various sections of this *Syllabus/Guide* may be used as models for examination questions or assignments.

The Course II mathematics syllabus is designed for a Regents level course culminating in the Three-Year Sequence for School Mathematics Course II Regents examination.

The Course II mathematics syllabus may be used as the basis of a local Course II for those students pursuing a local diploma. The local course may be less rigorous and should be followed by a local examination appropriate to the Course as modified by the local district.

Students with handicapping conditions are to be provided with appropriate alternative testing techniques as indicated on their Individualized Education Programs (IEPs). Such approved testing modifications should be used consistently in ongoing evaluations required for local and Regents diplomas.

# Resources for Teachers

(Joint Committee of the MAA and the NCTM.) *A Sourcebook of Applications of School Mathematics*. 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1980. 360 pp.

Provides a large number of realistic problems for use in the secondary school classroom. Includes answers and an annotated bibliography.

Maletsky, Evan, ed. and Yunker, Lee E. ed. *NCTM Student Math Notes*. 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1986. 21 pp.

Designed for use in junior high and secondary school instruction in mathematics. Includes five sets of activities for the students: 1) mathematics on postage stamps; historical procedures and mathematicians, metric conversion, geometric ideas, and formulas; 2) successful simulation, with activities on planning a supermarket, bowling, and random digits, and other suggestions for using a random digit table; 3) proof: geometric, algebraic, and numerical formulas; 4) golden rectangles and ratios with a physical model for generating them, directions for constructing a golden rectangle and for generating golden ratios via LOGO, and comments on the divine proportion; and, 5) primes, with the Sieve of Eratosthenes, twin primes, and other problems.

*Problem Solving in School Mathematics*. 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1980 Yearbook. 256 pp.

Views problem solving as a basic skill that can be learned and must be taught. Contains ideas, problems, examples, and illustrations for the classroom teacher for use at all levels of instruction, and an annotated bibliography.

*Resource Problems to Enhance the Teaching of Mathematics*. Ewing Hall, Department of Mathematical Sciences, Newark, DE 19716: University of Delaware. 1986. 300 pp.

Provides the teacher with a set of non-standard problems to be used in presenting in sections or discussions, or to be given to the students as challenging exercises for them to work on. For each problem, identifies a specific subject area and concept(s). Includes problems in arithmetic, algebra, geometry, coordinate geometry, trigonometry, and pre-calculus.

Sobel, Max A., ed. *Readings for Enrichment in Secondary School Mathematics*. 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1988. 298 pp.

Composed of articles from the *Mathematics Teacher*, *Enrichment Mathematics for High School* (NCTM's 28th Yearbook), and *Topics for Mathematics Clubs*, along with three new chapters on the harmonic mean, rotation matrices and complex numbers, and on the use of computers and calculators in performing arithmetic. Offers a variety of opportunities for expanding the mathematics curriculum for academically talented students.

*Spacemathematics: A Resource for Secondary School Teachers*. United States Government Printing Office, Washington, DC 20402: Superintendent of Documents. 1985. 172 pp.

Designed to serve as a supplement for high school level mathematics. Contains activities which aim to enhance mathematical knowledge and skills and to assist the students in understanding aerospace technology and its achievements. Includes chapters on "Mathematical Aspects of Some Recent NASA Missions," "Computation and Measurement," "Algebra," "Geometry," "Probability and Statistics," "Exponential and Logarithmic Functions," "Trigonometry," "Matrix Algebra," "Conic Sections," and "Calculus." In an appendix, contains an analysis of gravitational forces and the conic section trajectories, and a glossary.

*Teaching Statistics and Probability*. 1906 Association Drive, Reston, VA 22091: National Council of Teachers of Mathematics. 1981 Yearbook. 256 pp.

Deals with the place of statistics and probability in the curriculum, discusses how people use mathematics to cope with uncertainty, and provides ideas for classroom models, strategies, devices, activities, and projects.

## *Appendix A*

# STATEMENT OF REGENTS GOALS FOR ELEMENTARY AND SECONDARY SCHOOL STUDENTS – 1984

1. Each student will master communication and computation skills as a foundation to:
   - 1.1 Think logically and creatively.
   - 1.2 Apply reasoning skills to issues and problems.
   - 1.3 Comprehend written, spoken and visual presentations in various media.
   - 1.4 Speak, listen to, read, and write clearly and effectively in English.
   - 1.5 Perform basic mathematical calculations.
   - 1.6 Speak, listen to, read, and write at least one language other than English.
   - 1.7 Use current and developing technologies for academic and occupational pursuits.
   - 1.8 Determine what information is needed for particular purposes and be able to acquire, organize and use that information for those purposes.
2. Each student will learn methods of inquiry and knowledge gained through the following disciplines and use the methods and knowledge in interdisciplinary applications:
   - 2.1 English language and literature.
   - 2.2 History and social science.
   - 2.3 Mathematics.
   - 2.4 Natural sciences and technology.
   - 2.5 Language and literature in at least one language other than English.
3. Each student will acquire knowledge, understanding, and appreciation of the artistic, cultural, and intellectual accomplishments of civilization and develop the skills to express personal artistic talents. Areas include:
   - 3.1 Ways to develop knowledge and appreciation of the arts.
   - 3.2 Aesthetic judgments and the ability to apply them to works of art.
   - 3.3 Ability to use cultural resources of museums, libraries, theaters, historic sites, and performing arts groups.
   - 3.4 Ability to produce or perform works in at least one major art form.
   - 3.5 Materials, media and history of major art forms.
   - 3.6 Understanding of the diversity of cultural heritages.
4. Each student will acquire knowledge about political, economic, and social institutions and procedures in this country and other countries. Included are:
   - 4.1 Knowledge of American political, economic and social processes and policies at national, state, and local levels.
   - 4.2 Knowledge of political, economic, and social institutions and procedures in various nations; ability to compare the operation of such institutions; and understanding of the international interdependence of political, economic, social, cultural and environmental systems.
5. Each student will respect and practice basic civic values and acquire the skills, knowledge, understanding, and attitudes necessary to participate in democratic self-government. Included are:
   - 5.1 Understanding and acceptance of the values of justice, honesty, self-discipline, due process, equality, and majority rule with respect for minority rights.

5.2 Respect for self, others and property as integral to a self-governing, democratic society.

5.3 Ability to apply reasoning skills and process of democratic government to resolve societal problems and disputes.

6. Each student will develop the ability to understand and respect people of different race; sex; ability; cultural heritage; national origin; religion; and political, economic, and social background, and their values, beliefs and attitudes.

7. Each student will acquire knowledge of the ecological sequences of choices in the use of the environment and natural resources.

8. Each student will develop general career skills, attitudes, and work habits and make a self-assessment of career prospects. Students not directly pursuing postsecondary education will acquire entry-level employment skills.

9. Each student will learn knowledge, skills, and attitudes which enable development of:

9.1 Self-esteem.

9.2 The ability to maintain physical, mental and emotional health.

9.3 Understanding of the ill effects of alcohol, tobacco and other drugs.

10. Each student will develop a commitment to lifetime learning with the capacity for undertaking new studies, synthesizing new knowledge and experience with the known, and refining the ability to judge.

## *Appendix B*

## STUDENTS WITH HANDICAPPING CONDITIONS

The Board of Regents, through revising the Part 100 Regulations of the Commissioner and the Action Plan, has made a strong commitment to integrating the education of students with handicapping conditions into the total school program. According to Section 100.2(s) "Each student with a handicapping condition, as such term is defined in Section 200.1(ii) of this Chapter, shall have access to the full range of programs and services set forth in this Part to the extent that such programs and services are appropriate to such student's special educational needs." Districts must have policies and procedures in place to make sure that students with handicapping conditions have equal opportunities to access diploma credits, courses and requirements.

The majority of students with disabilities have the intellectual potential to master the curricula content requirements for a high school diploma. Most students who require special education attend regular education classes in conjunction with specialized instruction and/or related services. These students must attain the same academic standards as their nonhandicapped peers in order to meet these requirements. For this reason, it is very important that at all grade levels students with handicapping conditions receive instruction in the same content areas so as to receive the same informational base that will be required for proficiency on statewide testing programs and diploma requirements.

The teacher providing instruction through this syllabus/curriculum has the opportunity to provide an educational setting which will enable the students to explore their abilities and interests. Instruction may be provided to handicapped students either by teachers certified in this subject area or by special education teachers. Teachers certified in this subject area would be providing instruction to handicapped students who are recommended by the Committee on Special Education (CSE) as being able to benefit from instruction in a regular educational setting and are appropriately placed in this setting. Special education teachers may also provide this instruction to a class of students with handicapping conditions in a special class setting.

Teachers certified in the subject area should become aware of the needs of students with handicapping conditions participating in their classes. Instructional techniques and materials must be modified to the extent appropriate to provide students with handicapping conditions the opportunity to meet diploma requirements. Information or assistance is available through special education teachers, administrators, the Committee on Special Education or a student's Individualized Education Program (IEP).

1. Students with physical disabilities who use alternative communication devices may be restricted in their ability to do mathematical computations. For example, students who communicate with a head pointer and a memowriter (a calculator with adaptive large keys producing a narrow paper printout) are unable to type out the entire horizontal algorithm due to the small width of the printout paper. Horizontal equations which do not fit on one line are broken up on several different typed lines. In addition, when typing fractions, exponents, etc., each row of digits must be typed on separate lines. Scribes may be needed to individually assist these students during notetaking and computational activities.
2. Some students learn through different modalities, thus needing visual or auditory reinforcement of information that is presented. For instance, a student with a visual impairment may require verbal descriptions of board work and text materials; a student with a hearing impairment may need a detailed visual illustration with the board work and text material; or a student with a learning disability may need memory aids such as cue cards, arrows, numbering and/or index cards with directions on them.

3. Students have many different learning styles which may affect the manner in which they acquire information. For instance, some students may learn global concepts first and then learn algorithms, whereas some students remember routines but have difficulty understanding broader applications and concepts. A variety of presentation styles may be needed to accommodate these different learning styles.
4. Identify, define and preteach key vocabulary. Many terms in this syllabus are specific and may need continuous reinforcement for some students with handicapping conditions. It would also be helpful to provide a list of these key words to the special education teacher in order to provide additional reinforcement in the special educational setting.
5. Check periodically to determine student understanding of lectures, discussion, demonstrations, etc. and how this is related to the overall topic. Encourage students to express their understanding. It may be necessary to have small group discussions or work with a partner to determine this.
6. Provide students and special education teachers with a tape of lectures that contain substantial new vocabulary content and of guest speakers for further review within their special education class.
7. Assign a partner for the duration of a unit to a student as an additional resource to facilitate clarification of daily assignments, timelines for assignments, and access to daily class notes.
8. When assigning long-term projects/reports, provide a timeline with benchmarks as indicators for completion of major project/report sections. Students who have difficulty with organizational skills and time sequence, may need to see completion of sections to maintain the organization of a lengthy project/report.

Special education teachers providing this instruction must also become familiar with the goals and objectives of the curriculum. It is important that these teachers provide their students with the same or equivalent information contained in the curriculum.

Regardless of who provides the instruction, the cooperation between teachers of regular and of special education programs is essential. It is important for the students as well as the total school environment.

***Alternative Testing Techniques***

Another consideration in assisting students with handicapping conditions to meet the requirements of regular education is the use of alternative testing techniques. Alternative testing techniques are modifications of testing procedures or formats which provide students with handicapping conditions equal opportunity to participate in testing situations. Such techniques provide the opportunity to demonstrate mastery of skills and attainment of knowledge without being limited or unfairly restricted by the existence of a handicapping condition.

The Committee on Special Education (CSE) is responsible for identifying and documenting a student's need for alternative testing techniques. This determination is made when a student is initially referred to CSE, is reviewed annually for as long as the student receives special education services, and is reviewed when the student is determined to no longer need special education services. **The modifications are to be used consistently throughout the student's educational program.** Principals ensure that students who have been identified by CSE as educationally handicapped are provided with the alternative testing techniques which have been recommended by CSE and approved by the Board of Education.

**Alternative testing techniques which have been specified on student IEPs for use by a student must be used consistently in both special and regular education settings.** Regular classroom teachers should be aware of possible alternative testing techniques and should be skilled in their implementation.

The coordination and cooperation of the total school program will assist in providing the opportunity for a greater number of students with a handicapping condition to meet the requirements needed to pursue a high school diploma. The integrated provision of regular education programs, special education programs, remediation, alternative testing techniques, modified teacher techniques and materials, and access to credit through alternatives will assist in enabling such students to pursue high school diplomas to a greater degree. The teacher who provides instruction through this curriculum has a unique opportunity to assist such students in their individual goals.

Additional information on alternative testing modifications is available in the manual entitled *Alternative Techniques for Students with Handicapping Conditions*, which can be obtained from:

New York State Education Department
Office for Education of Children with
Handicapping Conditions
Room 1071 Education Building Annex
Albany, NY 12234

***Infusing Awareness of Persons with Disabilities Through Curriculum***

In keeping with the concept of integration, the following subgoal of the Action Plan was established:

*In all subject areas, revisions in the syllabi will include materials and activities related to* generic subgoals such as problem solving, reasoning skills, speaking, the capacity to search for information, the use of libraries, and increasing student awareness of and *information about the disabled*.

The purpose of this subgoal is to ensure that appropriate activities and materials are available to increase student awareness of disabilities and issues in regard to disabilities.

This curriculum, by design, includes information, activities and materials regarding persons with handicapping conditions. Teachers are encouraged to include other examples as may be appropriate to their classrooms or the situation at hand. Teachers are also encouraged to assess the classroom environment to determine how the environment may contribute to student awareness of persons with disabilities